

# SOME LOCAL ESTIMATES AND A UNIQUENESS RESULT FOR THE ENTIRE BIHARMONIC HEAT EQUATION

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**ABSTRACT.** We consider smooth solutions to the biharmonic heat equation on  $\mathbb{R}^n \times [0, T]$  for which the square of the Laplacian at time  $t$  is globally bounded from above by  $k_0/t$  for some  $k_0$  in  $\mathbb{R}^+$ , for all  $t \in [0, T]$ . We prove local, in space and time, estimates for such solutions. We explain how these estimates imply uniqueness of smooth solutions in this class.

## 1. INTRODUCTION

In this paper we prove local in space and time estimates for solutions  $u : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$  of the biharmonic heat flow,

$$(1.1) \quad \frac{\partial}{\partial t} u = -\Delta^2 u,$$

assuming that we have some global in time control on how the solution behaves as  $t \searrow 0$ . This control takes the form

$$(1.2) \quad t|\Delta u|^2(x, t) \leq k_0 < \infty$$

for  $t \in [0, T]$  for all  $x \in \mathbb{R}^n$  for some fixed  $k_0 \in \mathbb{R}^+$ . This means that it is possible for  $|\Delta u|^2(x, t)$  to approach infinity as  $t \searrow 0$ , but if it does so, then we have some control over the rate at which this occurs. Here  $\Delta u$  refers to the spatial Laplacian of  $u$ ,  $\Delta u(x, t) = \sum_{i=1}^n \nabla_i \nabla_i u(x, t)$  where  $\nabla_i u(x, t)$  is the partial derivative of  $u$  with respect to  $x_i$ .

The growth condition (1.2) is natural in the following senses. It is scale invariant: see the explanation of the scale invariance of  $b$  just after the definition of (A1) in Section 2. This behaviour also does occur in an asymptotic sense. That is, it is possible to construct a solution  $u \in C^\infty(\mathbb{R}^n \times (0, T))$  and to find points  $x(t) \in \mathbb{R}^n$  for all  $t > 0$  such that  $|\Delta u|^2(x(t), t) = \frac{k_0}{t}$  for some fixed  $k_0 > 0$  for all  $t > 0$ . We also find points  $y(t) \in \mathbb{R}^n$  for all  $t > 0$  such that  $(\Delta^2 u)(y(t), t) = \frac{k_0}{t}$  for some fixed  $k_0 \neq 0$  for all  $t > 0$ . That is, the speed of  $u$  is not integrable in time. See the example in Section 6.

Our first result is the following local estimate.

**Theorem 1.1.** *Let  $u : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$ ,  $T < \infty$ , be a smooth solution to (1.1) that satisfies*

$$(1.3) \quad |\Delta u|^2(x, t) \leq \frac{k_0}{t}$$

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for some  $k_0 \in \mathbb{R}$ , for all  $t \in [0, T]$ , all  $x \in \mathbb{R}^n$  and

$$\sup_{B_1(0)} \sum_{i=0}^{2n+2} |\nabla^i u_0|^2 \leq k_1 < \infty,$$

for some  $k_1 \in \mathbb{R}$ , where  $u_0(\cdot) := u(\cdot, 0)$ . Then there exists an  $N = N(n, k_0, k_1) > 0$  such that

$$|\Delta u|^2(x, t) \leq \frac{N}{(1 - |x|)^4}$$

for all  $x, t$  which satisfy  $x \in \overline{B}_1(0)$ ,  $(1 - |x|)^4 \geq Nt$ , and  $t \leq \frac{1}{N}$ ,  $t \leq T$ .

For  $i \in \mathbb{N}$  in the above and all that follows,  $\nabla^i u(x, t)$  refers to the full  $i$ -th spatial derivative, and  $|\nabla^i u(x, t)|$  the standard norm thereof. For example:  $\nabla^2 u(x, t) = (\nabla_i \nabla_j u(x, t))_{i,j \in \{1, \dots, n\}}$  and  $|\nabla^2 u(x, t)|^2 = \sum_{i,j=1}^n |\nabla_i \nabla_j u(x, t)|^2$ . The operator  $\Delta^k$  is defined by  $\Delta^k = (\Delta)^k$ .

If we have control on other derivatives as  $t \searrow 0$  then we obtain results on higher regularity.

**Theorem 1.2.** *Let  $s \in \mathbb{N}$ ,  $s \geq 2$  and  $u : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$ ,  $T < \infty$  be a smooth solution to (1.1) which satisfies*

$$(1.4) \quad \left( |\nabla^s u| + |\nabla^{s-1} u|^{s/(s-1)} + \dots + |\nabla u|^s \right)(x, t) \leq \frac{k_0}{t^{s/4}}$$

for some  $k_0 \in \mathbb{R}$ , for all  $t \in [0, T]$ , all  $x \in \mathbb{R}^n$  and

$$\sup_{B_1(0)} \sum_{i=0}^{2n+s+1} |\nabla^i u_0|^2 \leq k_1 < \infty,$$

for some  $k_1 \in \mathbb{R}$ . Then there exists an  $N = N(n, k_0, k_1, s) > 0$  such that

$$\left( |\nabla^s u| + |\nabla^{s-1} u|^{s/(s-1)} + \dots + |\nabla u|^s \right)(x, t) \leq \frac{N}{(1 - |x|)^s}$$

for all  $x, t$  which satisfy  $x \in \overline{B}_1(0)$ ,  $(1 - |x|)^4 \geq Nt$ , and  $t \leq \frac{1}{N}$ ,  $t \leq T$ .

In Section 5 we construct an example of a solution to (1.1) which starts off identically equal to zero, becomes immediately non-zero and is smooth in space and time. Solutions of this type for the heat equation are known to exist and were constructed by Tychonoff, see [T]. In particular, smooth solutions are not uniquely determined by their initial values:  $u(\cdot, \cdot) = 0$  is also a solution. If however we consider smooth solutions which satisfy (1.3) then the solution is uniquely determined by its initial value, as we show in Section 4.2. The theorem we prove is:

**Theorem 1.3.** *Let  $u, v : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$ ,  $T < \infty$  be smooth solutions to (1.1) which satisfy*

$$|\Delta v|^2(x, t) + |\Delta u|^2(x, t) \leq \frac{k_0}{t}$$

for some  $k_0 \in \mathbb{R}$ , for all  $t \in [0, T]$  and all  $x \in \mathbb{R}^n$  and

$$u_0(\cdot) = v_0(\cdot).$$

Then  $u \equiv v$ .

The uniqueness problem for the classical heat flow has a rich history. In the setting where one assumes the solution is non-negative, D. Widder established uniqueness for the heat flow on  $\mathbb{R}$  for solutions whose initial value is zero, see Theorem 5 of [W]. His method relied upon a specific integral representation of the solution, Theorem 4 of [W], which is valid for non-negative solutions.

This proved to be readily generalisable and so influential as to be given its own name: a uniqueness theorem is of Widder-type if the only hypotheses are on the geometry of the ambient space and that the solution be non-negative. For example, Aronson [Ar] proved that non-negative solutions to second order linear equations of divergence form (the coefficients of the operator being sufficiently regular) in  $\mathbb{R}^n$  are uniquely determined by their initial data: see Section 5 of that paper.

Here we take an approach reminiscent of [Si] that complements the existing literature. Our assumptions for Theorem 1.3 are global but we do not require any non-negativity of the solution. Although the flow (1.1) is higher-order, we are able to obtain our estimates using pointwise assumptions, as opposed to integral conditions.

The paper is organised as follows. Section 2 contains the proof of Theorem 1.1 and Theorem 1.2. These proofs require some energy estimates for solutions to (1.1), which is the subject of Section 3. Section 4 contains the proof of Theorem 1.3, and Section 5 contains full details on the Tychonoff-type solutions discussed above. We present in Section 6 details on the construction of the example mentioned above which shows that control of the form (1.2) is natural. We also show that the solution in this example has a speed which is not integrable.

Some of the estimates from Section 2 and Section 3 rely on interpolation inequalities which are not readily available in the current literature. These interpolation inequalities are proved in the Appendix.

## 2. A BLOWUP ARGUMENT

Let  $u : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$  be a smooth solution to (1.1). We consider the scale invariant quantity  $e(u) : B_1(0) \times [0, T] \rightarrow \mathbb{R}$  defined by

$$e(u)(x, t) := d^4(x) |\Delta u|^2(x, t),$$

where  $d(x) := (1 - |x|)$  is the distance from the boundary  $\partial B_1(0) = \{x \in \mathbb{R}^n \mid |x| = 1\}$  to the point  $x$  in  $B_1(0)$ . Note that  $e = 0$  on  $\partial B_1(0)$ . The function  $e$  is scale invariant in the following sense: If we define  $\tilde{u}(\tilde{x}, \tilde{t}) := u(c\tilde{x}, c^4\tilde{t}) - c_0$ , where  $c_0$  is an arbitrary constant in  $\mathbb{R}$ , then  $\tilde{u} : \mathbb{R}^n \times [0, \tilde{T}] \rightarrow \mathbb{R}$  is still a smooth solution to (1.1) and the quantity  $\tilde{e}(\tilde{x}, \tilde{t}) := \tilde{e}(\tilde{u})(\tilde{x}, \tilde{t})$  which is defined by

$$\tilde{e}(\tilde{u})(\tilde{x}, \tilde{t}) := \tilde{d}^4(\tilde{x}) |\Delta \tilde{u}|^2(\tilde{x}, \tilde{t}),$$

satisfies

$$\tilde{e}(\tilde{x}, \tilde{t}) := e(x, t),$$

where here  $x := c\tilde{x}$ ,  $t := c^4\tilde{t}$ ,  $\tilde{T} = \frac{T}{c^4}$ , and  $\tilde{d}(\tilde{x}) := (\frac{1}{c} - |\tilde{x}|)$  is the distance from  $x \in B_{1/c}(0)$  to  $\partial B_{1/c}(0)$ . The scale invariance of  $e$  can be seen as follows:

$$\begin{aligned} (\nabla \tilde{u})(\tilde{x}, \tilde{t}) &= c(\nabla u)(x, t), & \text{and hence } (\nabla^k \tilde{u})(\tilde{x}, \tilde{t}) &= c^k(\nabla^k u)(x, t), \\ \left(\frac{\partial}{\partial \tilde{t}} \tilde{u}\right)(\tilde{x}, \tilde{t}) &= c^4 \left(\frac{\partial}{\partial t} u\right)(x, t), & \text{and hence } \left(\left(\frac{\partial}{\partial \tilde{t}}\right)^k \tilde{u}\right)(\tilde{x}, \tilde{t}) &= c^{4k} \left(\left(\frac{\partial}{\partial t}\right)^k u\right)(x, t), \\ \tilde{d}(\tilde{x}) &= \left(\frac{1}{c} - |\tilde{x}|\right) \\ &= \frac{1}{c}(1 - |c\tilde{x}|) \\ &= \frac{1}{c}(1 - |x|) = \frac{1}{c}d(x), & \text{and hence } \tilde{d}^4(\tilde{x}) &= \frac{1}{c^4}d^4(x), \end{aligned}$$

where here  $(\frac{\partial}{\partial \tilde{t}})^k$  refers to  $k$  time derivatives, and  $\nabla^k u$  refers to  $k$  spatial derivatives, and we are assuming that  $k \in \mathbb{N}$  ( $k \neq 0$ ). Therefore

$$|\Delta \tilde{u}|^2(\tilde{x}, \tilde{t}) \tilde{d}^4(\tilde{x}) = |c^2 \Delta u|^2(x, t) \frac{1}{c^4} d^4(x) = |\Delta u|^2(x, t) d^4(x).$$

and  $e(x, t) = \tilde{e}(\tilde{x}, \tilde{t})$  as claimed. Note that

$$x \in B_1(0), d^4(x) \geq Nt \iff \tilde{x} \in B_{1/c}(0), \tilde{d}^4(\tilde{x}) \geq N\tilde{t}$$

in view of the definitions of the terms involved.

In the following, we will assume that

$$(A1) \quad b(u)(x, t) := t|\Delta u|^2(x, t) \leq k_0 < \infty \text{ for all } x \in \mathbb{R}^n, t \in [0, T],$$

for some fixed  $k_0 \in \mathbb{R}^+$ . That is, the quantity

$$(2.1) \quad Q(x, t) := Q(u)(x, t) := |\Delta u|^2(x, t)$$

may approach infinity as  $t \searrow 0$ , but it is only allowed to do so at a controlled, but non-integrable rate. Note that we have  $e(x, t) = d^4(x)Q(x, t)$  and  $b(x, t) = tQ(x, t)$ . The function  $b(x, t)$  is also scale invariant in the following sense: If we define  $\tilde{u}$ ,  $\tilde{x}$  and  $\tilde{t}$  as above, then  $\tilde{b}(\tilde{x}, \tilde{t}) := b(\tilde{u})(\tilde{x}, \tilde{t}) = b(x, t)$  and hence  $\tilde{b}(\tilde{x}, \tilde{t}) \leq k_0 < \infty$  for all  $\tilde{x} \in \mathbb{R}^n$  and all  $\tilde{t} \in [0, \tilde{T}]$ . The scale invariance of  $b$  may be verified with an argument similar to the one we used above to show that  $e$  is scale invariant. We are interested in the local behaviour of solutions  $u$  to (1.1) which satisfy (A1). In particular, if at time zero  $u_0 = u(\cdot, 0)$  satisfies

$$(A2) \quad \sup_{B_1(0)} \sum_{i=0}^{2n+2} |\nabla^i u_0|^2 \leq k_1 < \infty,$$

for some fixed  $k_1 \in \mathbb{R}^+$ , then we show that the solution satisfies estimates on a smaller ball for a short well-defined time interval. The following theorem is Theorem 1.1 of the introduction.

**Theorem 2.1.** *Let  $u : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$ ,  $T < \infty$  be a smooth solution to (1.1) which satisfies assumptions (A1) and (A2). Then there exists an  $N = N(n, k_0, k_1) > 0$  such that*

$$e(x, t) \leq N$$

for all  $x, t$  which satisfy  $x \in \overline{B}_1(0)$ ,  $d^4(x) \geq Nt$ , and  $t \leq \frac{1}{N}$ ,  $t \leq T$ .

For our theorem on higher order regularity, we modify the quantities above. Let  $s \in \mathbb{N}$ ,  $s \geq 2$  be given and fixed, and define

$$\begin{aligned} Q_s(u)(x, t) &= (|\nabla^s u| + |\nabla^{s-1} u|^{s/(s-1)} + \dots + |\nabla u|^s)^{4/s}(x, t) \\ e_s(u)(x, t) &= d^4(x) Q_s(u)(x, t) \\ b_s(u)(x, t) &= t Q_s(u)(x, t). \end{aligned}$$

These quantities are scale invariant in the sense explained above: for  $\tilde{u}$ ,  $\tilde{T}$ ,  $\tilde{t}$ ,  $\tilde{x}$  and  $\tilde{d}$  defined as above, and  $\tilde{Q}_s(\tilde{x}, \tilde{t}) := Q_s(\tilde{u})(\tilde{x}, \tilde{t})$  we have

$$\begin{aligned} \tilde{e}_s(\tilde{x}, \tilde{t}) &:= \tilde{d}^4(\tilde{x}) \tilde{Q}_s(\tilde{x}, \tilde{t}) = d^4(x) Q_s(u)(x, t), \quad \text{and} \\ \tilde{b}_s(\tilde{x}, \tilde{t}) &:= \tilde{t} \tilde{Q}_s(\tilde{x}, \tilde{t}) = t Q_s(u)(x, t). \end{aligned}$$

For this set-up we require

$$(A_s1) \quad b_s(u)(x, t) \leq k_0 < \infty$$

for all  $x \in \mathbb{R}^n$   $t \in [0, T]$ , and for some fixed  $k_0 \in \mathbb{R}^+$ ,  $s \geq 2$ ,  $s \in \mathbb{N}$ ; and

$$(A_s2) \quad \sup_{B_1(0)} \sum_{i=0}^{2n+s+1} |\nabla^i u_0|^2 \leq k_1 < \infty,$$

for some fixed  $k_1 \in \mathbb{R}^+$ . In this context we obtain the following variant of Theorem 2.1 above.

**Theorem 2.2.** *Let  $u : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$ ,  $T < \infty$  be a smooth solution to (1.1) which satisfies assumptions  $(A_s1)$  for some  $s \in \mathbb{N}$ ,  $s \geq 2$  and  $(A_s2)$ . Then there exists an  $N = N(n, k_0, k_1, s) > 0$  such that*

$$(2.2) \quad e_s(x, t) \leq N,$$

for all  $x, t$  which satisfy  $x \in \overline{B}_1(0)$ ,  $d^4(x) \geq Nt$ ,  $t \leq \frac{1}{N}$ ,  $t \leq T$ .

Note that this theorem is equivalent to Theorem 1.2 of the introduction. Under the same assumptions as Theorem 2.2, but with the condition that  $s \geq 4$ , we also obtain a local supremum bound for  $|u|$ :

**Corollary 2.3.** *Assume everything is as in Theorem 2.2 but that  $s \geq 4$ . Then we also have*

$$(2.3) \quad |u(x, t)| \leq \sqrt{k_1} + 1$$

for all  $x, t$  which satisfy  $x \in \overline{B}_1(0)$ ,  $d^4(x) \geq Nt$ ,  $t \leq \frac{1}{N}$ , where  $N$  is as in the conclusion of Theorem 2.2 above.

*Proof of Corollary 2.3.* Let  $(x, t_0)$  be a point which satisfies  $d^4(x) \geq Nt_0$  and  $t_0 \leq \frac{1}{N}$ . Then  $d^4(x) \geq Nt$  and  $t \leq \frac{1}{N}$  for all  $t \leq t_0$ . Hence, taking  $s = 4$  in Theorem 2.2 we see that  $|\Delta^2 u|(x, t) \leq \frac{N}{d^4(x)}$  for all  $t \leq t_0$  and that  $|u(x, 0)| \leq \sqrt{k_1}$  in view of  $(A_s2)$ . The evolution equation for  $u(x, t)$  is  $\frac{\partial}{\partial t} u(x, t) = -\Delta^2 u(x, t)$ . Integrating this from 0 to  $t_0$  and using the two estimates which we just derived, we see that  $|u(x, t_0)| \leq t_0 \frac{N}{d^4(x)} + \sqrt{k_1}$ . Using that  $d^4(x) \geq Nt_0$  we obtain the result.  $\square$

Now we prove Theorem 2.1.

*Proof of Theorem 2.1.* Define  $d$  resp.  $e$  to be zero on  $\mathbb{R}^n \setminus B_1(0)$  resp.  $(\mathbb{R}^n \setminus B_1(0)) \times [0, T]$ . Then  $e$  is continuous on  $\mathbb{R}^n \times [0, T]$ . Assume that the conclusion of the theorem is false, and let  $N \in \mathbb{N}$ . Note that  $e(x, 0) \leq k_0$  where  $k_0$  is the constant appearing in (A2). Without loss of generality  $N > k_0$ . The set of  $x \in \overline{B_1(0)}$ ,  $t \in [0, T]$  for which  $1 \geq d^4(x) \geq Nt$  and  $t \leq \frac{1}{N}$  is a compact set in  $\mathbb{R}^n \times [0, T]$  which we denote by  $K$ . By compactness of  $K$  and continuity of  $e$  and the fact that  $e(x, 0) \leq k_0 < N$  for all  $x \in \overline{B_1(0)}$ , there must be a first time  $t_0 \in (0, \frac{1}{N}]$  and (at least) one point  $x_0 \in B_1(0)$  such that  $e(x_0, t_0) = N$ . That is:  $e(x, t) < N$  for all  $(x, t) \in K$  with  $t < t_0$ , and  $e(x_0, t_0) = N$  for some point  $(x_0, t_0) \in K$ . Clearly we have  $d(x_0) > 0$  for such a point, that is,  $x_0 \in B_1(0)$ , since  $e(x_0, t_0) > 0$ . Rescale the solution  $u$  to  $\tilde{u}(\tilde{x}, \tilde{t}) := u(c\tilde{x}, c^4\tilde{t}) - c_0$ , where  $c_0 := u(c\tilde{x}_0, 0)$ , and  $c > 0$  is chosen so that  $\tilde{d}^4(\tilde{x}_0) = N$ . It is possible to choose  $c$  in this way:  $\tilde{d}(\tilde{x}) = \frac{1}{c}d(x)$ , so we choose  $c^4 = \frac{d^4(x_0)}{N}$ , which is larger than zero since  $d(x_0) > 0$  as we explained above. Our choice of  $c_0$  guarantees that  $\tilde{u}(\tilde{x}_0, 0) = 0$ . Note for later use that  $c^4 = \frac{d^4(x_0)}{N} \leq \frac{1}{N} (\leq 1)$ , and  $c \searrow 0$  as  $N \rightarrow \infty$ . Now

$$N = e(x_0, t_0) = \tilde{e}(\tilde{x}_0, \tilde{t}_0) = \tilde{d}^4(\tilde{x}_0)\tilde{Q}(\tilde{x}_0, \tilde{t}_0) = N\tilde{Q}(\tilde{x}_0, \tilde{t}_0)$$

due to scaling, and hence

$$\tilde{Q}(\tilde{x}_0, \tilde{t}_0) = 1.$$

Similarly,

$$N \geq e(x, t) = \tilde{e}(\tilde{x}, \tilde{t}) = \tilde{d}^4(\tilde{x})\tilde{Q}(\tilde{x}, \tilde{t})$$

for all  $(x, t) \in K$  with  $t \leq t_0$ , implies

$$(2.4) \quad \tilde{Q}(\tilde{x}, \tilde{t}) \leq \frac{N}{\tilde{d}^4(\tilde{x})}$$

for all  $\tilde{x} \in B_{1/c}(0)$  with  $\tilde{d}^4(\tilde{x}) \geq \tilde{t}N$  and  $\tilde{t} \leq \tilde{t}_0$ . Note that the inequality (2.4) is also valid for all  $\tilde{x}$  with  $\tilde{d}^4(\tilde{x}) \geq \tilde{t}N$  and  $\tilde{t} \leq \tilde{t}_0$ , since  $\tilde{d}(\tilde{x}) = 0$  outside of  $B_{1/c}(0)$  (here we define  $\frac{M}{0} = \infty$  for  $M > 0$ ). As in the paper [Si] we consider two cases: **Case 1**, where  $\tilde{d}^4(\tilde{x}_0) \geq 2N\tilde{t}_0$  (which is equivalent to  $\tilde{t}_0 \leq \frac{1}{2}$ , since  $\tilde{d}^4(\tilde{x}_0) = N$ ), and **Case 2**, where  $\tilde{d}^4(\tilde{x}_0) < 2N\tilde{t}_0$  (which is equivalent to  $1 \geq \tilde{t}_0 > \frac{1}{2}$ , since  $\tilde{t}_0N \leq \tilde{d}^4(\tilde{x}_0) < 2N\tilde{t}_0$  and  $\tilde{d}^4(\tilde{x}_0) = N$ ). Note that  $N\tilde{t}_0 \leq \tilde{d}^4(\tilde{x}_0)$  since  $(x_0, t_0) \in K$ ). We start with Case 1.

**Case 1:** In this case we have  $\tilde{d}^4(\tilde{x}_0) \geq 2N\tilde{t}_0$ . For  $\tilde{y}$  with  $\tilde{d}^4(\tilde{y}) \geq \frac{N}{2}$ , we obtain

$$(2.5) \quad \tilde{d}^4(\tilde{y}) \geq \frac{N}{2} \geq N\tilde{t}_0 \geq N\tilde{t}$$

for all  $\tilde{t} \leq \tilde{t}_0$ . Hence, we see that

$$(2.6) \quad \tilde{Q}(\tilde{y}, \tilde{t}) \leq \frac{N}{\tilde{d}^4(\tilde{y})} \leq 2$$

for all  $\tilde{t} \leq \tilde{t}_0$  in view of (2.5) and (2.4).

We also have that  $\tilde{d}^4(\tilde{x}_0) = N \geq \frac{N}{2}$  and so the above estimate also holds for  $\tilde{y} = \tilde{x}_0$  and  $\tilde{t} = \tilde{t}_0$ . We calculate

$$\begin{aligned} \frac{N}{2} &\leq \tilde{d}^4(\tilde{y}) = \left(\frac{1}{c} - |\tilde{y}|\right)^4 \\ \iff \left(\frac{1}{c} - |\tilde{y}|\right) &\geq \frac{N^{\frac{1}{4}}}{2^{\frac{1}{4}}} \\ (2.7) \quad \iff |\tilde{y}| &\leq -\frac{N^{\frac{1}{4}}}{2^{\frac{1}{4}}} + \frac{1}{c}. \end{aligned}$$

Furthermore  $\tilde{d}^4(\tilde{x}_0) = N$  implies  $|\tilde{x}_0| = -N^{\frac{1}{4}} + \frac{1}{c}$ . Assume that  $\tilde{y}$  is an arbitrary point with  $\tilde{y} \in B_{N^{\frac{1}{4}}/400}(\tilde{x}_0)$ . Then we have

$$\begin{aligned} |\tilde{y}| &\leq |\tilde{x}_0| + |\tilde{x}_0 - \tilde{y}| = -N^{\frac{1}{4}} + \frac{1}{c} + |\tilde{x}_0 - \tilde{y}| \\ &\leq -N^{\frac{1}{4}} + \frac{1}{c} + \frac{N^{\frac{1}{4}}}{400} \\ (2.8) \quad &\leq \frac{1}{c} - \frac{N^{\frac{1}{4}}}{2^{\frac{1}{4}}} \end{aligned}$$

and hence, in view of (2.7)

$$(2.9) \quad \tilde{d}^4(\tilde{y}) \geq \frac{N}{2}.$$

Therefore  $\tilde{y} \in B_{N^{\frac{1}{4}}/400}(\tilde{x}_0)$  and  $\tilde{t} \leq \tilde{t}_0 \leq \frac{1}{2}$  implies in Case 1 that

$$\tilde{Q}(\tilde{y}, \tilde{t}) \leq 2, \quad \text{and} \quad \tilde{Q}(\tilde{x}_0, \tilde{t}_0) = 1,$$

in view of (2.6) and the definition of  $\tilde{x}_0$  and  $\tilde{t}_0$ .

**Case 2.** In this case we have  $1 \geq \tilde{t}_0 > \frac{1}{2}$ . For all  $\tilde{t} \leq \frac{1}{2}$  and  $\tilde{y}$  with  $\tilde{d}^4(\tilde{y}) \geq \frac{N}{2}$  we have

$$\tilde{d}^4(\tilde{y}) \geq \frac{N}{2} \geq N\tilde{t}$$

and hence

$$(2.10) \quad \tilde{Q}(\tilde{y}, \tilde{t}) \leq \frac{N}{\tilde{d}^4(\tilde{y})} \leq 2$$

in view of (2.4). For  $\tilde{t}_0 \geq \tilde{t} \geq \frac{1}{2}$  we have

$$(2.11) \quad \tilde{Q}(\tilde{y}, \tilde{t}) \leq \frac{k_0}{\tilde{t}} \leq 2k_0,$$

in view of (A1). Note that we may assume without loss of generality that  $k_0 \geq 1$ . Now we know from (2.8) that  $y \in B_{N^{\frac{1}{4}}/400}(\tilde{x}_0)$  implies that  $\tilde{d}^4(\tilde{y}) \geq \frac{N}{2}$ . Hence, using the inequalities (2.10) and (2.11), we see that

$$(2.12) \quad \tilde{Q}(\tilde{y}, \tilde{t}) \leq 2k_0, \quad \text{and} \quad \tilde{Q}(\tilde{x}_0, \tilde{t}_0) = 1,$$

for all  $y \in B_{N^{\frac{1}{4}}/400}(\tilde{x}_0)$  and  $t \in [0, \tilde{t}_0]$ . We have shown that in both Case 1 and Case 2 we obtain the estimate (2.12). Now we use Corollary 3.5 to obtain a contradiction.

We use  $v : B_R(0) \times [0, \tilde{t}_0] \rightarrow \mathbb{R}$  to denote the rescaled solution  $\tilde{u} : B_{N^{\frac{1}{4}}/400}(\tilde{x}_0) \times [0, \tilde{t}_0] \rightarrow \mathbb{R}$ . That is  $v(\cdot, \cdot) = \tilde{u}(\cdot - \tilde{x}_0, \cdot)$ ,  $R = N^{\frac{1}{4}}/400$ ,  $\tilde{t}_0 \leq 1$ . The definition of  $\tilde{u}$  guarantees that  $\tilde{u}(\tilde{x}_0, 0) = 0$ , and hence we have  $v(0, 0) = 0$ . Also, using (A2) and the fact that  $c^4 \leq 1/N$  and  $N > 1$ , we see that

$$\begin{aligned}
 \sup_{B_{1/c}(0)} \sum_{i=1}^{2n+2} |\nabla^i v|^2(\cdot, 0) &= \sup_{B_1(x_0)} \sum_{i=1}^{2n+2} c^{2i} |\nabla^i u|^2(\cdot, 0) \\
 &\leq \sup_{B_1(x_0)} \sum_{i=1}^{2n+2} \frac{1}{N^{i/2}} |\nabla^i u(\cdot, 0)|^2 \\
 &\leq \frac{k_1}{N^{1/2}}.
 \end{aligned}
 \tag{2.13}$$

Hence, combining this estimate with the fact that  $v(0, 0) = 0$  and by choosing  $N$  sufficiently large, we may assume w.l.o.g. that

$$\sup_{B_1(0)} \sum_{i=0}^{2n+2} |\nabla^i v|^2(\tilde{x}, 0) \leq \tilde{\varepsilon}(N)
 \tag{2.14}$$

where  $\tilde{\varepsilon}(N) \rightarrow 0$  as  $N \rightarrow \infty$  ( $k_0, k_1, n$  are fixed in this argument). Define  $2\rho = R = N^{\frac{1}{4}}/400 \leq 1/c$  ( $c \leq \frac{1}{N^{1/4}}$  as we mentioned above) and  $p = p(n) = n + 1$ . Then  $\rho \rightarrow \infty$  as  $N \rightarrow \infty$ . Corollary 3.5 implies that

$$\frac{d}{dt} E_\eta^p(v) + E_\eta^{p+1}(v) \leq \frac{C}{\rho^{4p}} \int_{\mathbb{R}^n} |\Delta v|^2 \gamma^{s-4p}
 \tag{2.15}$$

for all  $t \leq \tilde{t}_0$  for all  $s > 4p + 4$ , where  $\eta = \gamma^s$ , and  $\gamma$  is a cutoff function as in  $(\gamma)$ , and  $C = C(n, s)$ . Choose  $s = 4p(n) + 5 = 4n + 9$ , so that  $C = C(n)$ . We know from the estimate (2.12) that  $Q(v) = |\Delta v|^2 \leq 2k_0$  on  $B_R(0) \times [0, \tilde{t}_0]$  and hence, combining this with (2.15) we have

$$\begin{aligned}
 \frac{d}{dt} E_\eta^p(v) + E_\eta^{p+1}(v) &\leq \frac{C}{\rho^{4p}} \int_{\mathbb{R}^n} |\Delta v|^2 \gamma^{s-4p} \\
 &\leq \frac{C}{\rho^{4p}} \int_{B_{2\rho}} 2k_0 \\
 &= 2\omega_n k_0 C \rho^{n-4p}
 \end{aligned}
 \tag{2.16}$$

which implies that

$$E_\eta^p(v)(t) \leq (2\omega_n k_0 C + \omega_n k_1) \rho^{n-4p} = (2\omega_n k_0 C + \omega_n k_1) \rho^{-3n-4}$$



for all  $t \leq \tilde{t}_0 \leq 1$  in view of the fact that

$$\begin{aligned}
E_\eta^p(v)(0) &= \int_{\mathbb{R}^n} |\Delta^p v_0|^2(\tilde{x}) \gamma^s d\tilde{x} \leq \int_{B_{2\rho}(0)} |\Delta^p v_0|^2(\tilde{x}) d\tilde{x} \\
&\leq \int_{B_{1/c}(0)} |\Delta^p v_0|^2(\tilde{x}) d\tilde{x} \\
&= \int_{B_{1/c}(0)} c^{4p} |\Delta^p u_0|^2(c\tilde{x}) d\tilde{x} \\
&= c^{4p-n} \int_{B_1(0)} |\Delta^p u_0|^2(x) dx \\
&\leq k_1 \omega_n c^{4p-n} \\
&\leq k_1 \omega_n \rho^{n-4p}
\end{aligned}$$

where we have used assumption (A2) again, the definition of  $v$ , the scaling properties of the derivatives of  $u(cx, 0)$ , and the fact that  $1/c \geq 2\rho \geq \rho$ . In particular,

$$\int_{B_1(0)} |\Delta^p v|^2 \leq (\omega_n 2k_0 C + k_1 \omega_n) \rho^{-3n-4} \rightarrow 0$$

as  $\rho \rightarrow \infty$ , in view of the fact that  $p = p(n) = n + 1$ . We have shown that

$$\int_{B_1(0)} |\Delta^p v|^2 \leq \varepsilon_p(k_0, k_1, n, \rho)$$

for all  $t \leq \tilde{t}_0 \leq 1$  where  $\varepsilon_p(k_0, k_1, n, \rho) \rightarrow 0$  as  $\rho \rightarrow \infty$ , that is, as  $N \rightarrow \infty$ . We can similarly show that

$$\int_{B_1(0)} |\Delta^{p-1} v|^2 \leq \varepsilon_{p-1}(k_0, k_1, n, \rho).$$

We also have

$$\begin{aligned}
\frac{d}{dt} \int_{B_1(0)} |\Delta^{p-2} v|^2 &= 2 \int_{B_1(0)} (\Delta^{p-2} v)(\Delta^p v) \\
&\leq \int_{B_1(0)} |\Delta^{p-2} v|^2 + \int_{B_1(0)} |\Delta^p v|^2 \\
&\leq \int_{B_1(0)} |\Delta^{p-2} v|^2 + \varepsilon_p
\end{aligned}$$

in view of Young's inequality, and the estimate just shown, and hence, after integrating in time from 0 to  $\tilde{t}_0 \leq 1$  we see that

$$(2.17) \quad \int_{B_1(0)} |\Delta^{p-2} v|^2 \leq \varepsilon_{p-2}(N)$$

for all  $t \in [0, \tilde{t}_0 \leq 1]$  with  $\varepsilon_{p-2}(N) \rightarrow 0$  as  $N \rightarrow \infty$ : we leave out dependence on  $k_1, k_0, n$  since these variables are fixed. More explicitly:  $f(t) := e^{-t} (\int_{B_1(0)} |\Delta^{p-2} v|^2)(t) - 2t\varepsilon_p$  satisfies  $\frac{df}{dt}(t) \leq 0$  for all  $0 \leq t \leq \tilde{t}_0$  and  $f(0) = (\int_{B_1(0)} |\Delta^{p-2} v|^2)(0) \leq \tilde{\varepsilon}(N) \rightarrow 0$  as  $N \rightarrow \infty$  in view of (2.14), and so, integrating  $f$  from 0 to  $t_0$ , we see that the estimate (2.17) is true.

Continuing in this way, we get, for  $N$  sufficiently large:

$$(2.18) \quad \int_{B_1(0)} |\Delta^l v|^2 \leq \varepsilon_l(N)$$

for  $l = p, p-2, p-4, \dots, 1$  (or 0). Starting with  $p-1$  instead of  $p$ , we similarly get

$$(2.19) \quad \int_{B_1(0)} |\Delta^l v|^2 \leq \varepsilon_l(N)$$

for  $l = p-1, p-3, p-5, \dots, 1$  (or 0), where  $\varepsilon_l(N) \rightarrow 0$  as  $N \rightarrow \infty$ . That is

$$(2.20) \quad \int_{B_1(0)} |\Delta^l v|^2 \leq \varepsilon_l(N)$$

for  $l \in \{0, \dots, p\}$ , where  $\varepsilon_l(N) \rightarrow 0$  as  $N \rightarrow \infty$ .

Using the  $L^2$  estimates, Lemma 7.1 from the Appendix, and the estimate (2.20) we see that

$$(2.21) \quad \int_{B_{1/2}(0)} |\nabla^l v|^2 \leq \hat{\varepsilon}(N)$$

for all  $0 \leq l \leq 2p = 2n+2$ , where  $\hat{\varepsilon}(N) \rightarrow 0$  as  $N \rightarrow \infty$  (choose  $\sigma = \sigma(n) = \frac{1}{4p(n)}$ , so that  $1 - 2p\sigma = 1/2$ ).

Applying the Sobolev-Morrey inequality [E, Theorem 6, Section 5.6.3], with  $p, k$  there equal to 2, and  $2n+2$  respectively, we see that

$$\tilde{Q}(0, t_0) = |\Delta v|^2(0, t_0) \leq C(n) \left( \sum_{l=0}^{2n+2} \int_{B_{1/2}(0)} |\nabla^l v|^2(\cdot, t_0) \right)^{\frac{1}{2}} \leq C(2n+3)^{\frac{1}{2}} (\hat{\varepsilon})^{\frac{1}{2}}$$

and hence

$$\tilde{Q}(0, t_0) \rightarrow 0$$

as  $N \rightarrow \infty$ . This contradicts the fact that  $\tilde{Q}(0, t_0) = 1$ . □

The proof of Theorem 2.2 is essentially the same as the proof of Theorem 2.1.

*Proof of Theorem 2.2.* Replace  $Q(x, t)$  by  $Q_s(x, t)$ ,  $e(x, t)$  by  $e_s(x, t)$ ,  $b(x, t)$  by  $b_s(x, t)$ , and  $Q(u)$  by  $Q_s(u)$  and repeat the above proof. At the point where  $|\Delta v|^2 = Q(v) \leq 2k_0$  on  $B_R(0) \times [0, t_0]$  is used in the inequality (2.16), use instead the fact that  $|\Delta v|^2 \leq |\nabla^2 v|^2 \leq Q_s(v) \leq 2k_0$ . Also choose  $p(n) = n + (s/2)$  or  $p = n + \frac{(s+1)}{2}$  in the proof: whichever is an integer. The last part of the proof, where Morrey's embedding Theorem is used, has to be slightly modified:  $Q_s(v)(0, t_0) = 1$  implies that  $|\nabla^r v|(0, t_0) \geq \delta(s) > 0$  for some  $r \in \{1, \dots, s\}$  for some small  $\delta(s) > 0$ : otherwise the sum of the terms appearing in  $Q_s(v)(0, t_0)$  would be less than one.

Applying the Sobolev-Morrey inequality [E, Theorem 6, Section 5.6.3] with  $p, k$  there equal to 2,  $2n+s$  respectively, we see that

$$\begin{aligned} 0 < \delta^2(s) &\leq |\nabla^r v|^2(0, t_0) \\ &\leq C(n, s) \left( \sum_{l=0}^{2n+s} \int_{B_{1/2}(0)} |\nabla^l v|^2(\cdot, t_0) \right)^{1/2} \\ &\leq C(2n+1+s)^{\frac{1}{2}} (\hat{\varepsilon}(N))^{1/2}, \end{aligned}$$

which leads to a contradiction if  $N$  is chosen large enough, since  $\hat{\varepsilon}(N) \rightarrow 0$  as  $N \rightarrow \infty$ . □

## 3. A PRIORI ENERGY ESTIMATES

In this section we shall prove some estimates for the weighted energies

$$(3.1) \quad E_\eta^k(u) = \int_{\mathbb{R}^n} |\Delta^k u|^2 \eta, \quad \eta \in C_{loc}^\infty(\mathbb{R}^n), \quad \text{supp } \eta \subset \subset \mathbb{R}^n,$$

where  $k \in \mathbb{N}_0$  and  $u : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$  is a smooth solution to (1.1). In the above equation and in that which follows all integrals are with respect to Lebesgue measure. Note that  $E_\eta^k$  are all finite for any  $k \in \mathbb{N}_0$  and  $t \in [0, T]$  since  $u : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$  is smooth. The purpose of the a priori estimates in this section is to quantify how global quantities such as the various Sobolev norms of the solution behave along the flow (1.1).

**Lemma 3.1.** *Let  $u : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$  be a smooth solution to (1.1). For all  $t \in [0, T]$ ,*

$$(3.2) \quad \begin{aligned} & \frac{d}{dt} E_\eta^k(u) + 2E_\eta^{k+1}(u) \\ &= -2 \int_{\mathbb{R}^n} (\Delta^{k+1} u)(\nabla_i \Delta^k u)(\nabla_i \eta) + 2 \int_{\mathbb{R}^n} (\Delta^k u)(\nabla_i \Delta^{k+1} u)(\nabla_i \eta). \end{aligned}$$

for all  $k \in \mathbb{N}_0$ .

*Proof.* Differentiating,

$$\begin{aligned} \frac{d}{dt} E_\eta^k(u) &= 2 \int_{\mathbb{R}^n} (\Delta^k u)(-\Delta^{k+2} u) \eta \\ &= 2 \int_{\mathbb{R}^n} (\nabla_i \Delta^k u)(\nabla_i \Delta^{k+1} u) \eta + 2 \int_{\mathbb{R}^n} (\Delta^k u)(\nabla_i \Delta^{k+1} u)(\nabla_i \eta) \\ &= -2 \int_{\mathbb{R}^n} |\Delta^{k+1} u|^2 \eta - 2 \int_{\mathbb{R}^n} (\nabla_i \Delta^k u)(\nabla_i \eta)(\Delta^{k+1} u) \\ &\quad + 2 \int_{\mathbb{R}^n} (\Delta^k u)(\nabla_i \Delta^{k+1} u)(\nabla_i \eta) \\ &= -2E_\eta^{k+1}(u) \\ &\quad - 2 \int_{\mathbb{R}^n} (\Delta^{k+1} u)(\nabla_i \Delta^k u)(\nabla_i \eta) + 2 \int_{\mathbb{R}^n} (\Delta^k u)(\nabla_i \Delta^{k+1} u)(\nabla_i \eta). \end{aligned}$$

Rearranging gives the lemma.  $\square$

We now specialise by setting  $\eta = \gamma^s$ ,  $s > 0$  to be chosen, and  $\gamma \in C_{loc}^\infty(\mathbb{R}^n)$  satisfying

$$(\gamma) \quad \chi_{B_\rho(0)} \leq \gamma \leq \chi_{B_{2\rho}(0)}, \quad \rho > 0, \quad \text{and} \quad |\nabla \gamma| \leq \frac{c_\gamma}{\rho}, \quad |\nabla^2 \gamma| \leq \frac{c_\gamma}{\rho^2},$$

where  $c_\gamma \geq 1$  is an absolute constant depending only on  $n$ .

In the following proofs we make extensive use of the elementary inequality

$$(3.3) \quad ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$$

for  $a, b$  real numbers and  $\varepsilon > 0$ .

**Lemma 3.2.** *Let  $u \in C_{loc}^\infty(\mathbb{R}^n)$ . Suppose  $\eta = \gamma^s$ ,  $s > 8$ , and  $\gamma$ ,  $c_\gamma$  are as in  $(\gamma)$ . Then for any  $\varepsilon > 0$  and for all  $k \in \mathbb{N}$  we have*

$$-2 \int_{\mathbb{R}^n} (\Delta^{k+1}u)(\nabla_i \Delta^k u)(\nabla_i \eta) \leq \varepsilon E_\eta^{k+1}(u) + \frac{c_1(\varepsilon, s, n)}{\rho^8} \int_{\mathbb{R}^n} |\Delta^{k-1}u|^2 \gamma^{s-8},$$

where  $c_1(\varepsilon, s, n) < \infty$  is a constant depending on  $\varepsilon, s$  and  $n$ .

*Proof.* Throughout the proof  $\delta_i$  denote positive parameters to be chosen. Using (3.3) and  $(\gamma)$ ,

$$\begin{aligned} -2 \int_{\mathbb{R}^n} (\Delta^{k+1}u)(\nabla_i \Delta^k u)(\nabla_i \eta) &= -2s \int_{\mathbb{R}^n} (\Delta^{k+1}u)(\nabla_i \Delta^k u)(\nabla_i \gamma) \gamma^{s-1} \\ (3.4) \qquad \qquad \qquad &\leq \delta_1 E_\eta^{k+1}(u) + \frac{c_\gamma^2 s^2}{\delta_1 \rho^2} \int_{\mathbb{R}^n} |\nabla \Delta^k u|^2 \gamma^{s-2}. \end{aligned}$$

Now,

$$\begin{aligned} &\int_{\mathbb{R}^n} |\nabla \Delta^k u|^2 \gamma^{s-2} \\ &= - \int_{\mathbb{R}^n} (\Delta^k u)(\Delta^{k+1}u) \gamma^{s-2} - (s-2) \int_{\mathbb{R}^n} (\Delta^k u)(\nabla_i \Delta^k u)(\nabla_i \gamma) \gamma^{s-3} \\ &\leq \rho^2 \delta_2 \int_{\mathbb{R}^n} |\Delta^{k+1}u|^2 \gamma^s + \frac{1}{4\delta_2 \rho^2} \int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla \Delta^k u|^2 \gamma^{s-2} + \frac{c_\gamma^2 (s-2)^2}{2\rho^2} \int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} \\ &= \rho^2 \delta_2 \int_{\mathbb{R}^n} |\Delta^{k+1}u|^2 \gamma^s + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla \Delta^k u|^2 \gamma^{s-2} \\ &\quad + \frac{1}{2\rho^2} \left( \frac{1}{2\delta_2} + c_\gamma^2 (s-2)^2 \right) \int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} \end{aligned}$$

Absorbing the second term on the right into the left we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} |\nabla \Delta^k u|^2 \gamma^{s-2} \\ (3.5) \qquad \qquad \qquad &\leq 2\rho^2 \delta_2 \int_{\mathbb{R}^n} |\Delta^{k+1}u|^2 \gamma^s + \frac{1}{\rho^2} \left( \frac{1}{2\delta_2} + c_\gamma^2 (s-2)^2 \right) \int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4}. \end{aligned}$$

We now need an interpolation inequality. From Lemma 7.2 we know that

$$\begin{aligned} &\frac{1}{\rho^2} \int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} \\ &\leq \delta_3 \rho^2 \int_{\mathbb{R}^n} |\Delta^{k+1}u|^2 \gamma^s + \frac{c_{\delta_3}}{\rho^6} \int_{\mathbb{R}^n} |\Delta^{k-1}u|^2 \gamma^{s-8}, \end{aligned}$$

where  $c_{\delta_3} = c_{\delta_3}(\delta_3, s, n) = 2^4 \left( \frac{1}{\delta_3} + 2^9 s^4 c_\gamma^4 \right)$ . Using this in (3.5) we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla \Delta^k u|^2 \gamma^{s-2} &\leq 2\rho^2 \delta_2 \int_{\mathbb{R}^n} |\Delta^{k+1} u|^2 \gamma^s \\ &\quad + \left( \frac{1}{2\delta_2} + c_\gamma^2 (s-2)^2 \right) \left( \delta_3 \rho^2 \int_{\mathbb{R}^n} |\Delta^{k+1} u|^2 \gamma^s + \frac{c_{\delta_3}}{\rho^6} \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8} \right) \\ &= \left( 2\rho^2 \delta_2 + \delta_3 \rho^2 \left( \frac{1}{2\delta_2} + c_\gamma^2 (s-2)^2 \right) \right) \int_{\mathbb{R}^n} |\Delta^{k+1} u|^2 \gamma^s \\ &\quad + \frac{c_{\delta_3}}{\rho^6} \left( \frac{1}{2\delta_2} + c_\gamma^2 (s-2)^2 \right) \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8} \end{aligned}$$

and hence

$$\begin{aligned} \frac{1}{\rho^2} \int_{\mathbb{R}^n} |\nabla \Delta^k u|^2 \gamma^{s-2} &\leq \left( \delta_3 \left( \frac{1}{2\delta_2} + c_\gamma^2 (s-2)^2 \right) + 2\delta_2 \right) \int_{\mathbb{R}^n} |\Delta^{k+1} u|^2 \gamma^s \\ (3.6) \quad &\quad + \frac{c_{\delta_3}}{\rho^8} \left( \frac{1}{2\delta_2} + c_\gamma^2 (s-2)^2 \right) \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8}. \end{aligned}$$

Combining (3.6) with (3.4) gives

$$\begin{aligned} -2 \int_{\mathbb{R}^n} (\Delta^{k+1} u)(\nabla_i \Delta^k u)(\nabla_i \eta) \\ \leq \left[ \delta_1 + \frac{c_\gamma^2 s^2}{\delta_1} \left( \delta_3 \left( \frac{1}{2\delta_2} + c_\gamma^2 (s-2)^2 \right) + 2\delta_2 \right) \right] E_\eta^{k+1}(u) \\ \quad + \frac{c_\gamma^2 s^2}{\delta_1} \frac{c_{\delta_3}}{\rho^8} \left( \frac{1}{2\delta_2} + c_\gamma^2 (s-2)^2 \right) \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8}. \end{aligned}$$

We choose  $\delta_i = \delta_i(s, \varepsilon, n) > 0$  so that

$$\left[ \delta_1 + \frac{c_\gamma^2 s^2}{\delta_1} \left( \delta_3 \left( \frac{1}{2\delta_2} + c_\gamma^2 (s-2)^2 \right) + 2\delta_2 \right) \right] \leq \varepsilon.$$

This can be achieved by the choice

$$\delta_1 = \frac{\varepsilon}{4}, \quad \delta_2 = \frac{\varepsilon^2}{32c_\gamma^2 s^2}, \quad \delta_3 = \frac{\varepsilon^4}{8c_\gamma^4 s^2(16s^2 + \varepsilon^2(s-2)^2)}$$

for example. Recalling the definition of  $c_{\delta_3} = c(\delta_3, s, n) = 2^4 \left( \frac{1}{\delta_3} + 2^9 s^4 c_\gamma^4 \right)$  yields the result.  $\square$

**Lemma 3.3.** *Let  $u \in C_{loc}^\infty(\mathbb{R}^n)$ . Suppose  $\eta = \gamma^s$ ,  $s > 8$ , and  $\gamma, c_\gamma$  are as in  $(\gamma)$ . Then for any  $\varepsilon > 0$  and for all  $k \in \mathbb{N}$  we have*

$$2 \int_{\mathbb{R}^n} (\Delta^k u)(\nabla_i \Delta^{k+1} u)(\nabla_i \eta) \leq \varepsilon E_\eta^{k+1}(u) + \frac{c_2(\varepsilon, s, n)}{\rho^8} \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8},$$

where  $c_2(\varepsilon, s, n) < \infty$  is a constant depending on  $\varepsilon, s, n$ .

*Proof.* Again, throughout the proof  $\delta_i > 0$  denote positive parameters to be chosen. Integrating by parts,

$$\begin{aligned} 2 \int_{\mathbb{R}^n} (\Delta^k u)(\nabla_i \Delta^{k+1} u)(\nabla_i \eta) &= -2 \int_{\mathbb{R}^n} (\nabla_i \Delta^k u)(\Delta^{k+1} u)(\nabla_i \eta) \\ (3.7) \quad &\quad - 2 \int_{\mathbb{R}^n} (\Delta^k u)(\Delta^{k+1} u)(\Delta \eta) \end{aligned}$$

Lemma 3.2 deals with the first term:

$$(3.8) \quad -2 \int_{\mathbb{R}^n} (\nabla_i \Delta^k u)(\Delta^{k+1} u)(\nabla_i \eta) \leq \frac{\varepsilon}{2} E_\eta^{k+1}(u) + \frac{c_1(\frac{\varepsilon}{2}, s, n)}{\rho^8} \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8}.$$

Since  $\Delta \eta = s\gamma^{s-1} \Delta \gamma + s(s-1)\gamma^{s-2} |\nabla \gamma|^2$  we have

$$(3.9) \quad \begin{aligned} & -2 \int_{\mathbb{R}^n} (\Delta^k u)(\Delta^{k+1} u)(\Delta \eta) \\ & \leq 2\delta_1 \int_{\mathbb{R}^n} |\Delta^{k+1} u|^2 \eta + \frac{1}{\delta_1 \rho^4} \left( c_\gamma^2 s^2 + c_\gamma^4 s^2 (s-1)^2 \right) \int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4}, \end{aligned}$$

where we used the fact that  $\gamma^{s-2}(\cdot) \leq \gamma^{s-4}(\cdot)$ , which is true since  $0 \leq \gamma(\cdot) \leq 1$  on  $\mathbb{R}^n$ . Lemma 7.2 yields the estimate

$$\begin{aligned} & \frac{1}{\delta_1 \rho^4} \left( c_\gamma^2 s^2 + c_\gamma^4 s^2 (s-1)^2 \right) \int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} \\ & \leq \frac{\delta_2}{\delta_1} \left( c_\gamma^2 s^2 + c_\gamma^4 s^2 (s-1)^2 \right) \int_{\mathbb{R}^n} |\Delta^{k+1} u|^2 \gamma^s \\ & \quad + \frac{c_{\delta_2}}{\delta_1 \rho^8} \left( c_\gamma^2 s^2 + c_\gamma^4 s^2 (s-1)^2 \right) \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8} \end{aligned}$$

where  $c_{\delta_2} = c(\delta_2, s, n) = 2^4 \left( \frac{1}{\delta_2} + 2^9 s^4 c_\gamma^4 \right)$ . Combining this with (3.9) we get

$$(3.10) \quad \begin{aligned} & -2 \int_{\mathbb{R}^n} (\Delta^k u)(\Delta^{k+1} u)(\Delta \eta) \leq \left( 2\delta_1 + \frac{\delta_2}{\delta_1} \left( c_\gamma^2 s^2 + c_\gamma^4 s^2 (s-1)^2 \right) \right) \int_{\mathbb{R}^n} |\Delta^{k+1} u|^2 \eta \\ & \quad + \frac{c_{\delta_2}}{\delta_1 \rho^8} \left( c_\gamma^2 s^2 + c_\gamma^4 s^2 (s-1)^2 \right) \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8}. \end{aligned}$$

Combining (3.10) with (3.7)-(3.8) and choosing  $\delta_i = \delta_i(\varepsilon, n, s) > 0$  so that

$$\frac{\delta_2}{\delta_1} \left( c_\gamma^2 s^2 + c_\gamma^4 s^2 (s-1)^2 \right) + 2\delta_1 = \varepsilon/2$$

yields the result. One possible choice is

$$\delta_1 = \frac{\varepsilon}{16}, \quad \delta_2 = \frac{\varepsilon^2}{128 c_\gamma^2 \left( s^2 + c_\gamma^2 s^2 (s-1)^2 \right)}.$$

□

**Corollary 3.4.** *Let  $u : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$  be a smooth solution to (1.1). Suppose  $\eta = \gamma^s$ ,  $s > 8$ , and  $\gamma$ ,  $c_\gamma$ , are as in  $(\gamma)$ , and  $k \in \mathbb{N}$ . Then*

$$\frac{d}{dt} E_\eta^k(u) + \frac{3}{2} E_\eta^{k+1}(u) \leq \frac{c_3(n, s)}{\rho^8} \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8} dx,$$

where  $c_3(n, s)$  is constant depending only on  $n$  and  $s$ .

*Proof.* We combine Lemmata 3.1–3.3 as follows. Lemma 3.1 states that

$$(3.11) \quad \begin{aligned} & \frac{d}{dt} E_\eta^k(u) + 2E_\eta^{k+1}(u) \\ & = -2 \int_{\mathbb{R}^n} (\Delta^{k+1} u)(\nabla_i \Delta^k u)(\nabla_i \eta) + 2 \int_{\mathbb{R}^n} (\Delta^k u)(\nabla_i \Delta^{k+1} u)(\nabla_i \eta). \end{aligned}$$

The two terms on the right hand side are estimated by Lemma 3.2 and Lemma 3.3 respectively. Adding together the estimates we find, for any  $\varepsilon_1, \varepsilon_2 > 0$ ,

$$\begin{aligned} & -2 \int_{\mathbb{R}^n} (\Delta^{k+1}u)(\nabla_i \Delta^k u)(\nabla_i \eta) + 2 \int_{\mathbb{R}^n} (\Delta^k u)(\nabla_i \Delta^{k+1}u)(\nabla_i \eta) \\ & \leq (\varepsilon_1 + \varepsilon_2) E_\eta^{k+1}(u) + \frac{c_1(\varepsilon_1, n, s) + c_2(\varepsilon_2, n, s)}{\rho^8} \int_{\mathbb{R}^n} |\Delta^{k-1}u|^2 \gamma^{s-8}. \end{aligned}$$

In particular choosing  $\varepsilon_i = \frac{1}{4}$  and combining this with (3.11) we have

$$\frac{d}{dt} E_\eta^k(u) + 2E_\eta^{k+1}(u) \leq \frac{1}{2} E_\eta^{k+1}(u) + \frac{c_3}{2\rho^8} \int_{\mathbb{R}^n} |\Delta^{k-1}u|^2 \gamma^{s-8},$$

where  $c_3$  is a constant depending only on  $s$  and  $n$ . Absorbing the first term on the right into the left yields the claimed estimate.  $\square$

**Corollary 3.5.** *Let  $u : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$  be a smooth solution to (1.1). Suppose  $k \in \mathbb{N}$ ,  $\eta = \gamma^s$ ,  $s > 4k$ , where  $\gamma, c_\gamma$  are as in  $(\gamma)$ . Then*

$$\frac{d}{dt} E_\eta^k(u) + E_\eta^{k+1}(u) \leq \frac{c_4(n, s)}{\rho^{4k}} \int_{\mathbb{R}^n} |\Delta u|^2 \gamma^{s-4k},$$

where  $c_4(n, s)$  is a constant depending only on  $n$  and  $s$ .

*Proof.* We first consider the case where  $k = 1$ . Using Lemma 3.1 and integration by parts we find

$$\begin{aligned} & \frac{d}{dt} E_\eta^1(u) + 2E_\eta^2(u) \\ & = -2 \int_{\mathbb{R}^n} (\Delta^2 u)(\nabla_i \Delta u)(\nabla_i \eta) + 2 \int_{\mathbb{R}^n} (\Delta u)(\nabla_i \Delta^2 u)(\nabla_i \eta) \\ (3.12) \quad & = -4 \int_{\mathbb{R}^n} (\Delta^2 u)(\nabla_i \Delta u)(\nabla_i \eta) - 2 \int_{\mathbb{R}^n} (\Delta u)(\Delta^2 u)(\Delta \eta). \end{aligned}$$

We claim that

$$(3.13) \quad -4 \int_{\mathbb{R}^n} (\Delta^2 u)(\nabla_i \Delta u)(\nabla_i \eta) \leq \varepsilon E_\eta^2(u) + \frac{c(\varepsilon, n, s)}{\rho^4} \int_{\mathbb{R}^n} |\Delta u|^2 \gamma^{s-4},$$

and

$$(3.14) \quad -2 \int_{\mathbb{R}^n} (\Delta u)(\Delta^2 u)(\Delta \eta) \leq \varepsilon E_\eta^2(u) + \frac{c(\varepsilon, n, s)}{\rho^4} \int_{\mathbb{R}^n} |\Delta u|^2 \gamma^{s-4},$$

hold. Given the above estimates, we may conclude the required statement for the case  $k = 1$  as follows. Choosing  $\varepsilon = \frac{1}{2}$  in each of (3.13), (3.14) and combining with (3.12) we find

$$\frac{d}{dt} E_\eta^1(u) + 2E_\eta^2(u) \leq E_\eta^2(u) + \frac{c(n, s)}{\rho^4} \int_{\mathbb{R}^n} |\Delta u|^2 \gamma^{s-4},$$

whereupon subtraction of  $E_\eta^2(u)$  from both sides yields the desired estimate.

The estimate (3.14) is (3.9) with  $\delta_1 = \varepsilon/2$  and  $k = 1$ . It remains to prove the estimate (3.13). We compute

$$(3.15) \quad -4 \int_{\mathbb{R}^n} (\Delta^2 u)(\nabla_i \Delta u)(\nabla_i \eta) \leq \delta_1 E_\eta^2(u) + \frac{4c_\gamma^2 s^2}{\delta_1 \rho^2} \int_{\mathbb{R}^n} |\nabla \Delta u|^2 \gamma^{s-2}.$$

Now estimate

$$\begin{aligned}
& \int_{\mathbb{R}^n} |\nabla \Delta u|^2 \gamma^{s-2} \\
&= - \int_{\mathbb{R}^n} (\Delta u)(\Delta^2 u) \gamma^{s-2} - (s-2) \int_{\mathbb{R}^n} (\Delta u)(\nabla_i \Delta u)(\nabla_i \gamma) \gamma^{s-3} \\
&\leq \rho^2 \delta_2 \int_{\mathbb{R}^n} |\Delta^2 u|^2 \gamma^s + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla \Delta u|^2 \gamma^{s-2} \\
&\quad \frac{1}{2\rho^2} \left( \frac{1}{2\delta_2} + c_\gamma^2 (s-2)^2 \right) \int_{\mathbb{R}^n} |\Delta u|^2 \gamma^{s-4}.
\end{aligned}$$

Absorbing the second term on the right into the left we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^n} |\nabla \Delta u|^2 \gamma^{s-2} \\
(3.16) \quad & \leq 2\rho^2 \delta_2 \int_{\mathbb{R}^n} |\Delta^2 u|^2 \gamma^s + \frac{1}{\rho^2} \left( \frac{1}{2\delta_2} + c_\gamma^2 (s-2)^2 \right) \int_{\mathbb{R}^n} |\Delta u|^2 \gamma^{s-4}.
\end{aligned}$$

Combining (3.15) with (3.16) we find

$$\begin{aligned}
& -4 \int_{\mathbb{R}^n} (\Delta^2 u)(\nabla_i \Delta u)(\nabla_i \eta) \\
& \leq \left( \delta_1 + 2\rho^2 \delta_2 \frac{4c_\gamma^2 s^2}{\delta_1 \rho^2} \right) E_\eta^2(u) + \frac{4c_\gamma^2 s^2}{\delta_1 \rho^2} \left( \frac{1}{\rho^2} \left( \frac{1}{2\delta_2} + c_\gamma^2 (s-2)^2 \right) \right) \int_{\mathbb{R}^n} |\Delta u|^2 \gamma^{s-4}.
\end{aligned}$$

Choosing  $\delta_1 > 0$  and  $\delta_2 > 0$  such that  $\left( \delta_1 + 2\delta_2 \frac{4c_\gamma^2 s^2}{\delta_1} \right) \leq \varepsilon$  yields (3.13).

Let us now continue by considering the case where  $k \geq 2$ . In this case we have  $k-1 \in \mathbb{N}$ , and  $s > 4k$  implies  $s-4 > 4k-4 = 4(k-1)$ . Using these facts and Corollary 7.3, we see that

$$(3.17) \quad \frac{1}{\rho^8} \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8} \leq \frac{1}{\rho^4} \int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} + \frac{c(s, n)}{\rho^{4k}} \int_{\mathbb{R}^n} |\Delta u|^2 \gamma^{s-4k}.$$

and, using Corollary 7.3 again,

$$(3.18) \quad \frac{1}{\rho^4} \int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} \leq \delta_1 \int_{\mathbb{R}^n} |\Delta^{k+1} u|^2 \gamma^s + \frac{c(\delta_1, s, n)}{\rho^{4k}} \int_{\mathbb{R}^n} |\Delta u|^2 \gamma^{s-4k}.$$

Combining (3.17) with (3.18) and then choosing  $\delta_1 = \frac{1}{2c_3}$ , where  $c_3(n, s)$  is as in the previous Corollary, yields

$$(3.19) \quad \frac{c_3}{\rho^8} \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8} \leq \frac{1}{2} \int_{\mathbb{R}^n} |\Delta^{k+1} u|^2 \gamma^{s-4} + \frac{\tilde{c}}{\rho^{4k}} \int_{\mathbb{R}^n} |\Delta u|^2 \gamma^{s-4k},$$

for some  $\tilde{c} = \tilde{c}(n, s)$ . Using (3.19) to estimate the right hand side of Corollary 3.4 we find

$$\begin{aligned}
\frac{d}{dt} E_\eta^k(u) + \frac{3}{2} E_\eta^{k+1}(u) &\leq \frac{c_3}{\rho^8} \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8} \\
&\leq \frac{1}{2} \int_{\mathbb{R}^n} |\Delta^{k+1} u|^2 \gamma^{s-4} + \frac{\tilde{c}}{\rho^{4k}} \int_{\mathbb{R}^n} |\Delta u|^2 \gamma^{s-4k},
\end{aligned}$$

which, after absorbing the first term on the right into the left, becomes

$$\frac{d}{dt} E_\eta^k(u) + E_\eta^{k+1}(u) \leq \frac{c_4}{\rho^{4k}} \int_{\mathbb{R}^n} |\Delta u|^2 \gamma^{s-4k}$$



as required.  $\square$

#### 4. UNIQUENESS

In this section we prove that smooth solutions to (1.1) which satisfy  $|\Delta u|^2(\cdot, t) \leq \frac{k_0}{t}$  are uniquely determined by their initial values.

**Theorem 4.1.** *Let  $v : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$ ,  $T < \infty$  be a smooth solution to (1.1) which satisfies*

$$(4.1) \quad |\Delta v|^2(x, t) \leq \frac{k_0}{t}$$

for some  $k_0 \in \mathbb{R}$ , for all  $t \in [0, T]$  and all  $x \in \mathbb{R}^n$  and

$$(4.2) \quad v_0 \equiv 0.$$

Then  $v \equiv 0$ .

*Proof.* Since

$$\sup_{B_1(0)} \sum_{i=0}^p |\nabla^i v_0|^2 = 0$$

for any  $p > 0$ , (c.f. (A2)), Theorem 2.1 tells us that  $|\Delta v|^2(0, t) \leq 2N(n, k_0)$  for some  $N = N(n, k_0) \in \mathbb{R}$  for all  $t \leq 1/N$ . By setting  $\tilde{v}(\cdot, t) = v(\cdot - x_0, t)$  and using Theorem 2.1 for  $\tilde{v}$ , we see that  $|\Delta v|^2(x_0, t) \leq N(n, k_0)$  for all  $t \leq 1/N$ , for all  $x_0 \in \mathbb{R}^n$ . Corollary 3.5 implies that

$$(4.3) \quad \frac{\partial}{\partial t} E_\eta^p(v) + E_\eta^{p+1}(v) \leq 2CN\omega_n \rho^{n-4p} = 2CN\omega_n \rho^{-3n-4}$$

where  $p$  is now fixed and chosen to be  $p(n) = n + 1$ , and  $\eta$  is a non-negative cutoff function with  $\eta = 1$  on  $B_\rho(0)$ , and  $C = C(n)$ . To see this repeat the argument from the inequality (2.15) up to (2.16) but use this  $v$  (instead of the  $v$  appearing there) and use the fact that  $k_0 = 0$ , and  $|\Delta v|^2 \leq N$  for all  $t \leq 1/N$  for this  $v$ . This implies that  $E_\eta^p(v)(t) \leq 2CN\omega_n \rho^{-3n-4}$  for all  $t \leq 1/N \leq 1$ , since  $E_\eta^p(v)$  is non-negative, and  $E_\eta^p(v)(0) = 0$ . Letting  $\rho \rightarrow \infty$ , we see that  $\int_{\mathbb{R}^n} |\Delta^p v|^2 = 0$  for all  $t \leq 1/N$ . Similarly  $\int_{\mathbb{R}^n} |\Delta^{p-1} v|^2 = 0$  for all  $t \leq 1/N$ . Now use

$$\begin{aligned} \frac{\partial}{\partial t} \int_{B_1(0)} |\Delta^{p-2} v|^2 &= 2 \int_{B_1(0)} \Delta^p v \Delta^{p-2} v \\ &\leq \int_{B_1(0)} |\Delta^p v|^2 + |\Delta^{p-2} v|^2 = \int_{B_1(0)} |\Delta^{p-2} v|^2 \end{aligned}$$

which tells us, after integrating, that  $\int_{B_1(0)} |\Delta^{p-2} v|^2 = 0$  for all  $t \leq 1/N$ . Differentiating  $\int_{B_1(0)} |\Delta^{p-3} v|^2$  w.r.t. time and using  $\int_{B_1(0)} |\Delta^{p-1} v|^2 = 0$  we obtain, using the same argument, that  $\int_{B_1(0)} |\Delta^{p-3} v|^2 = 0$  for all  $t \leq 1/N$ . Continuing in this way, we find that  $\int_{B_1(0)} |\Delta^l v|^2 = 0$  for all  $0 \leq l \leq p$ , for all  $t \leq 1/N$ . Similarly, we obtain  $\int_{B_1(x_0)} |\Delta^l v|^2 = 0$  for all  $0 \leq l \leq p$ , for all  $t \leq 1/N$  for all  $x_0 \in \mathbb{R}^n$ . In particular, by choosing  $l = 0$ , we see that  $v(\cdot, t) = 0$  for all  $t \leq 1/N$ ,  $t \leq T$ . Repeating this argument for the function  $\tilde{v}(\cdot, \tilde{t}) = v(\cdot, \tilde{t} + 1/N)$ , we see that  $v(\cdot, t) = 0$  for all  $t \leq T$ , as required.  $\square$

**Corollary 4.2.** *Let  $u, v : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$ ,  $T < \infty$  be smooth solutions to (1.1) which satisfy*

$$(4.4) \quad |\Delta v|^2(x, t) + |\Delta u|^2(x, t) \leq \frac{k_0}{t}$$

*for all  $t \in [0, T]$  and all  $x \in \mathbb{R}^n$  and*

$$(4.5) \quad u_0(\cdot) = v_0(\cdot).$$

*Then  $u \equiv v$ .*

*Proof.* Set  $s = u - v$ . Then  $s$  satisfies the conditions of Theorem 4.1. Hence  $s \equiv 0$  as required.  $\square$

## 5. A TYCHONOFF-TYPE SOLUTION AND NON-UNIQUENESS

In this section we describe a simple modification to the classical Tychonoff counterexample, see [T], which establishes non-uniqueness for complete solutions of the polyharmonic heat equation. We follow the construction given in [J, Chapter 7, Section 1 (a), pp 211–213].

Let  $k \in \mathbb{N}$  and consider a solution  $u : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$  to

$$(5.1) \quad (\partial_t - \Delta^k)u = 0 \quad \text{on } \mathbb{R}^n \times [0, T],$$

$$(5.2) \quad u(\cdot, 0) = u_0(\cdot) \quad \text{on } \mathbb{R}^n.$$

We shall construct infinitely many solutions to (5.1)-(5.2) which have zero as their initial data.

For functions  $g_j : [0, T] \rightarrow \mathbb{R}$  to be chosen, set

$$u(x, t) = \sum_{j=0}^{\infty} g_j(t) x_1^{2jk}.$$

The convergence of this series will be guaranteed by our choice of  $g_j$ , and verified later. Differentiating formally, we find

$$\begin{aligned} \sum_{j=0}^{\infty} (\partial_t g_j)(t) x_1^{2jk} &= \partial_t u(x, t) = \Delta^k u(x, t) \\ &= \sum_{j=1}^{\infty} (2jk)(2jk-1) \cdots (2jk-2k+1) g_j(t) x_1^{2jk-2k} \\ &= \sum_{j=0}^{\infty} (2jk+2k)(2jk+2k-1) \cdots (2jk+1) g_{j+1}(t) x_1^{2jk} \end{aligned}$$

for all  $j \in \mathbb{N}_0$ . We are thus led to the recurrence relation

$$(5.3) \quad (\partial_t g_j) = (2jk+2k)(2jk+2k-1) \cdots (2jk+1) g_{j+1}$$

for all  $j \in \mathbb{N}_0$ . We set  $g_j(t) = \lambda(j, k) g_0^{(j)}(t)$ , where  $(g_0)^j$  refers to  $j$  temporal derivatives of  $g_0$ , and  $\lambda(j, k)$  is a constant to be determined depending only on  $j, k$ . Using this choice of  $g_j$ , we see that (5.3) is satisfied, provided that

$$g_0^{(j+1)} \lambda(j, k) = (2jk+2k)(2jk+2k-1) \cdots (2jk+1) \lambda(j+1, k) g_0^{(j+1)},$$

which for  $g_0^{(j+1)} \neq 0$  is equivalent to

$$\frac{\lambda(j, k)}{\lambda(j+1, k)} = (2jk + 2k)(2jk + 2k - 1) \cdots (2jk + 1).$$

Using  $\lambda(0, k) = 1$ , we see that this implies that  $\lambda(j, k) = \frac{1}{(2jk)!}$  for all  $j \in \mathbb{N}_0$  (we use  $0! := 1$ ). Let us now set

$$g_0(t) = \exp(-t^{-p})$$

for  $t > 0$  and  $p > 1$ .

**Lemma 5.1.** *There is an absolute constant  $\varepsilon_0 > 0$  and a  $p > 1$  such that the estimate*

$$\left| g_0^{(j)}(t) \right| \leq \frac{j! 2^j}{t^j} \exp(-\varepsilon_0 (2t)^{-p})$$

holds for all  $t > 0$ .

*Proof.* Consider the function  $h(z) = \exp(-z^{-p})$  for  $p > 1$ . Since  $z^p := \exp(p \operatorname{Log}(z))$  is analytic on  $\mathbb{C} \setminus (-\infty, 0]$ ,  $h$  is analytic on  $\mathbb{C} \setminus (-\infty, 0]$ : for polar coordinates  $z = re^{i\theta}$  with  $\theta \in (-\pi, \pi)$  we are using  $\operatorname{Log}(z) := \log(r) + i\theta$ , and hence  $z^p = r^p e^{ip\theta}$ . For  $0 < \rho < t$ , Cauchy's integral formula on  $S_\rho(t + 0i) = S_\rho(t)$ , the circle in  $\mathbb{C}$  with radius  $\rho$  centred at  $t + 0i$ , gives

$$g_0^{(j)}(t) = h^{(j)}(t + 0i) = \frac{j!}{2\pi i} \int_{S_\rho(t)} \frac{h(z)}{(z - t)^{j+1}} dz.$$

This gives the estimate

$$(5.4) \quad |g_0^{(j)}(t)| \leq \frac{j!}{\rho^j} \sup_{z \in S_\rho(t)} |h(z)|.$$

In polar coordinates  $z = r \exp(i\theta)$ ,  $\theta \in (-\pi, \pi)$  we have

$$h(z) = \exp(-r^{-p} e^{-ip\theta}) = \exp(-r^{-p}(\cos(p\theta) - i \sin(p\theta))).$$

Therefore

$$(5.5) \quad |h(z)| \leq \exp(-r^{-p} \cos p\theta).$$

For  $z \in S_\rho(t)$ , we have

$$-\frac{\pi}{2} < -\theta_0 \leq \theta \leq \tan^{-1}(\rho/\sqrt{t^2 - \rho^2}) =: \theta_0 < \frac{\pi}{2}$$

Note that  $\theta_0$  doesn't depend on  $p$ . So we may choose  $p > 1$  such that  $p\theta_0 < \frac{\pi}{2}$ : this is possible since  $\theta_0 < \frac{\pi}{2}$ . We then have  $\cos(p\theta) \geq \cos(p\theta_0) =: \varepsilon_0 > 0$  for all  $\theta \in (-\theta_0, \theta_0)$ . Since  $r \leq 2t$  we may estimate

$$-r^{-p} \cos p\theta \leq -\varepsilon_0 (2t)^{-p}$$

for all  $\theta \in (-\theta_0, \theta_0)$ , which combined with our earlier estimate (5.5) yields

$$\sup_{z \in S_\rho(t)} |h(z)| \leq \exp(-\varepsilon_0 (2t)^{-p}).$$

Inserting this into the estimate (5.4) and choosing  $\rho = t/2$  finishes the proof.  $\square$

Lemma 5.1 implies

$$\begin{aligned}
|u(x, t)| &\leq \sum_{j=0}^{\infty} |g_j(t)| |x_1|^{2jk} \\
&= \sum_{j=0}^{\infty} \frac{|g_0^{(j)}(t)|}{(2jk)!} |x_1|^{2jk} \\
&\leq \sum_{j=0}^{\infty} \frac{j! 2^j}{t^j (2jk)!} |x_1|^{2jk} \exp(-\varepsilon_0 (2t)^{-p}) \\
&\leq \exp(-\varepsilon_0 (2t)^{-p}) \sum_{j=0}^{\infty} \frac{j!}{(2jk)!} \left(\frac{|x_1|^{2k}}{t/2}\right)^j \\
&\leq \exp(-\varepsilon_0 (2t)^{-p}) \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{|x_1|^{2k}}{t/2}\right)^j \\
&= \exp\left(-\frac{\varepsilon_0}{(2t)^p} + \frac{|x_1|^{2k}}{t/2}\right).
\end{aligned}$$

Here we have used  $\frac{j!}{(2j)!} \leq \frac{1}{j!}$  for all  $j \in \mathbb{N}_0$  which may be seen using induction. Therefore  $u$  is well-defined for every  $t > 0$ . Moreover,  $p > 1$  implies that the first term above always dominates for small  $t$  and so  $u$  converges uniformly to zero on compact subsets of  $\mathbb{R}^n$  as  $t \searrow 0$ . More precisely, let  $K$  be a compact subset of  $\mathbb{R}^n$  with diameter  $d$  and  $0 \in K$ . Then  $|x| \leq d$  and for  $x \in K$

$$\lim_{t \searrow 0} |u|(x, t) \leq \lim_{t \searrow 0} \exp\left(-\frac{\varepsilon_0}{(2t)^p} + \frac{d^{2k}}{t/2}\right) = 0.$$

A similar argument shows that all derivatives of  $u$  exist and converge uniformly to zero on compact subsets of  $\mathbb{R}^n$  as  $t \searrow 0$ . We explain this in the following. Assuming  $x = x_1$  satisfies  $|x| \leq d$  where  $d \geq 1$  and taking  $s$  spatial derivatives formally we find

$$\begin{aligned}
|(\partial_x)^s u(x, t)| &= \left| \sum_{j \geq s/(2k)} (g_0)^j(t)(x)^{2jk-s} \frac{(2jk)(2jk-1) \dots (2jk-s+1)}{(2jk)!} \right| \\
&\leq \sum_{j \geq s/(2k)} |(g_0)^j(t)(x)^{2jk-s}| \left| \frac{(2jk)(2jk-1) \dots (2jk-s+1)}{(2jk)!} \right| \\
&\leq \sum_{j \geq s/(2k)} |g_0^j(t)| d^{2jk-s} \frac{1}{(2jk-s)!} \\
&\leq \sum_{j \geq s/(2k)} |g_0^j(t)| (|d|^{2k})^j \frac{1}{(2jk-s)!} \\
&\leq \exp(-\varepsilon_0 (2t)^{-p}) \sum_{j \geq s/(2k)} (2d^{2k}/t)^j \frac{j!}{(2jk-s)!} \\
&\leq \exp(-\varepsilon_0 (2t)^{-p}) \sum_{\{j \mid 2kj \geq s\}} (2d^{2k}/t)^j \frac{(kj)!}{(2jk-s)!}
\end{aligned}$$

$$\begin{aligned}
&\leq s! \exp(-\varepsilon_0(2t)^{-p}) \sum_{\{j \mid 2kj \geq s\}} \frac{(2d^{2k}/t)^j}{kj!} \\
&\leq s! \exp(-\varepsilon_0(2t)^{-p}) \sum_{\{j \mid 2kj \geq s\}} \frac{(2d^{2k}/t)^j}{j!} \\
&\leq s! \exp(-\varepsilon_0(2t)^{-p}) \exp(2d^{2k}/t),
\end{aligned}$$

which goes to zero as  $t \searrow 0$ . Here we used that  $\frac{(r!)^2}{(2r-s)!} \leq s!$  for all  $r \geq s, r, s \in \mathbb{N}_0$ , which may be verified using induction on  $r$ . Since  $s$  time derivatives of  $u$  are formally given by  $2ks$  spatial derivatives of  $u$ , we see that all mixed derivatives (space and time) of  $u$  exist for  $t > 0$  and converge uniformly on (spatial) compact sets  $K \subset \mathbb{R}^n$  to 0.

By extending  $u$  to be zero for all  $t \leq 0$  we have a solution  $u \in C^\infty(\mathbb{R}^n \times (-\infty, \infty))$  to (5.1)-(5.2) which is non-zero for  $t > 0$  and satisfies  $u \equiv 0$  for all  $t \leq 0$ .

## 6. AN EXAMPLE

Let  $u_0 : \mathbb{R}^n \rightarrow \mathbb{R}$  be given by  $u_0(x_1, x_2, \dots, x_n) := 1$  if  $x_1 > 0$ ,  $u_0(x_1, x_2, \dots, x_n) := 0$  if  $x_1 \leq 0$ . Setting  $u(x, t) := \int_{\mathbb{R}^n} u_0(x - y)b(y, t)dy$ , with  $b : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$  the bi-harmonic heat kernel on  $\mathbb{R}^n$ , we see that the function  $u : \mathbb{R}^n \times (0, T) \rightarrow \mathbb{R}$  is smooth and solves  $\frac{\partial}{\partial t}u(x, t) = -\Delta^2 u(x, t)$  for all  $t > 0$  for all  $x \in \mathbb{R}^n$ , and that  $u(\cdot, t) \rightarrow u_0(\cdot)$  uniformly on any compact set  $K$  contained in  $\mathbb{R}^n \setminus \{x \in \mathbb{R}^n \mid x_1 = 0\}$ . Furthermore, there exists a  $k_0 > 0$  such that for all  $s > 0$  there exists a  $x_s \in \mathbb{R}^n$  such that  $|\Delta u|^2(x_s, s) = \frac{k_0}{s}$ . The biharmonic heat kernel  $b$  is given by

$$b(x, t) = (2\pi)^{-n/2} t^{-n/4} \int_{\mathbb{R}^n} e^{i\langle w, x \rangle t^{-1/4} - |w|^4} dw.$$

We verify of all these facts below.

We have (see the Appendix of [KL], and the papers [GP], [GG1],[GG2] for further, related and similar estimates)

$$(6.1) \quad \left| \left( \frac{\partial}{\partial t} \right)^l (\nabla^k) b(x, t) \right| \leq C(k, l, m) (t^{-p(k,l)+m/4} + t^{(m-1)/4}) |x|^{-m}$$

for all  $l, k \in \mathbb{N}_0, m \in \mathbb{N}_0$ , for some  $p(k, l) \in \mathbb{N}$  for all  $x \in \mathbb{R}^n$  for all  $t > 0$ . This can be seen as follows. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be  $f(y) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\langle w, y \rangle - |w|^4} dw$ , so that  $b(x, t) = t^{-n/4} f(-\frac{x}{t^{1/4}})$ . Then  $f$  is the Fourier transform of the function  $l : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $l(w) := e^{-|w|^4}$  which is in  $\mathcal{S}$ , the so called Schwartz Space (see [SW] Chapter I.3 where this set of functions is defined and called the space of *testing functions*). Hence  $f$  itself is in  $\mathcal{S}$  (see [SW], Theorem 3.2 Chapter I.3), in particular  $|\nabla_\alpha f|(x) \leq \frac{c(|\alpha|, m)}{|x|^m}$  for any  $m \in \mathbb{N}_0$  and any multi-index  $\alpha = (\alpha_1, \dots, \alpha_n)$ :  $\alpha_i \in \mathbb{N}_0$  for all  $i = 1, \dots, n$ ,  $|\alpha| := \alpha_1 + \alpha_2 + \dots + \alpha_n$ , and we have used the notation  $\nabla_\alpha f := \nabla_{\alpha_1} \nabla_{\alpha_2} \dots \nabla_{\alpha_n} f$ .

Using the representation  $b(x, t) = t^{-n/4} f(-\frac{x}{t^{1/4}})$  and the fact that  $f$  is in  $\mathcal{S}$  we get

$$\begin{aligned} \left| \left( \frac{\partial}{\partial t} \right)^l (\nabla^k) b(x, t) \right| &\leq (t^{-p(k, l)} + t^{-1/4}) \sum_{0 \leq |\alpha| \leq k+l} |\nabla_\alpha f| \left( -\frac{x}{t^{1/4}} \right) \\ &\leq (t^{-p(k, l)} + t^{-1/4}) \frac{c(k, l, m)}{|x/t^{1/4}|^m} \\ &= c(k, l, m) \frac{t^{-p(k, l)+m/4} + t^{(m-1)/4}}{|x|^m} \end{aligned}$$

which proves the estimate (6.1) since  $m \in \mathbb{N}_0$  was arbitrary. This shows that the function  $u(x, t) := \int_{\mathbb{R}^n} u_0(x-y) b(y, t) dy = \int_{\mathbb{R}^n} u_0(z) b(x-z, t) dz$  is well defined for any measurable  $L^\infty$  function  $u_0 : \mathbb{R}^n \rightarrow \mathbb{R}$  for all  $t > 0$ , and is differentiable in time and space for all  $t > 0$  for all  $x \in \mathbb{R}^n$  and the derivative is given by differentiating under the integral sign (in view of the Lebesgue dominated convergence Theorem):

$$\left( \frac{\partial}{\partial t} \right)^l (\nabla^k) u(x, t) = \int_{\mathbb{R}^n} u_0(z) \left( \left( \frac{\partial}{\partial t} \right)^l (\nabla^k) b \right) (x-z, t) dz.$$

Using the fact that  $\frac{\partial}{\partial t} b = -\Delta^2 b$  (see below for an explanation), we get  $u : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$  is smooth and satisfies  $\frac{\partial}{\partial t} u = -\Delta^2 u$ . Notice also that  $\int_{\mathbb{R}^n} b(x, t) dx = \int_{\mathbb{R}^n} t^{-n/4} f(-\frac{x}{t^{1/4}}) dx = \int_{\mathbb{R}^n} f(-z) dz = \int_{\mathbb{R}^n} b(z, 1) dz = 1$  (the last equality is explained below). Hence, for  $x \in B_\varepsilon(z)$  where  $z = (z_1, \dots, z_n)$  has  $z_1 > 2\varepsilon$ , we have

$$\begin{aligned} |u(x, t) - 1| &= \left| \int_{\mathbb{R}^n} b(x-y, t) (u_0(y) - 1) dy \right| \\ (6.2) \quad &= \left| \int_{B_\varepsilon(x)} b(x-y, t) (u_0(y) - 1) dy \right. \\ &\quad \left. + \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} b(x-y, t) (u_0(y) - 1) dy \right| \\ &= 0 + \left| \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} b(x-y, t) (u_0(y) - 1) dy \right| \\ &\leq \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} 2|b(x-y, t)| dy \\ &\leq \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{c(m, n) t^{m-n/4}}{|x-y|^{4m}} dy \\ &\leq C(\varepsilon, m, n) t^{m-n/4} \\ (6.3) \quad &\leq C(\varepsilon, m, n) t^2 \end{aligned}$$

for  $m > 2 + n/4$  for  $t \leq 1$  which goes to zero as  $t \rightarrow 0$ . Similarly  $|u(x, t)| \leq C(\varepsilon, m, n) t^2$  goes to zero for all  $x \in B_\varepsilon(z)$  where  $z = (z_1, \dots, z_n)$  has  $z_1 < -2\varepsilon$ . Hence  $u(\cdot, t) \rightarrow u_0$  uniformly on compact sets  $K \subseteq \mathbb{R}^n \setminus \{x \in \mathbb{R}^n \mid x_1 = 0\}$ . The definition of  $u_0$  and  $u$  guarantees that  $u(cx, c^4 t) = u(x, t)$  for all  $c, t > 0$ . We verify

this now. Notice first that

$$\begin{aligned} b(cx, c^4t) &= (2\pi)^{-n/2} (c^4t)^{-n/4} \int_{\mathbb{R}^n} e^{i(c^4t)^{-1/4} \langle w, cx \rangle - |w|^4} dw \\ (6.4) \quad &= c^{-n} (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{it^{-1/4} \langle w, x \rangle - |w|^4} dw \end{aligned}$$

that is  $b(cx, c^4t) = c^{-n}b(x, t)$  for all  $x \in \mathbb{R}^n$  for all  $c > 0$ . Also, the definition of  $u_0$  guarantees that  $u_0(cz) = u_0(z)$  for all  $z \in \mathbb{R}^n$  and all  $c > 0$ . Making a change of variable  $y = cw$  in the definition of  $u$ , and then using  $b(cw, c^4t) = c^{-n}b(w, t)$  and the property of  $u_0$  just mentioned, we calculate

$$\begin{aligned} u(cx, c^4t) &:= \int_{\mathbb{R}^n} u_0(cx - y) b(y, c^4t) dy = \int_{\mathbb{R}^n} u_0(cx - cw) b(cw, c^4t) c^n dw \\ (6.5) \quad &= \int_{\mathbb{R}^n} u_0(x - w) b(w, t) dw \\ &= u(x, t). \end{aligned}$$

There must exist a point  $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}^+$  with  $\Delta u(x_0, t_0) \neq 0$ : if not, then  $\frac{\partial}{\partial t} u = -\Delta^2 u = 0$  for all  $t > 0$ , and hence  $u(x, t) = u(x, s)$  for all  $0 < s < t$ , and hence, using  $u \rightarrow u_0$  on  $\mathbb{R}^n \setminus \{x \in \mathbb{R}^n | x_1 = 0\}$  as  $t \searrow 0$  as explained above, we have  $u(x, t) = 1$  on  $\mathbb{R}^n \cap \{x \in \mathbb{R}^n | x_1 > 0\}$ ,  $u(x, t) = 0$  on  $\mathbb{R}^n \cap \{x \in \mathbb{R}^n | x_1 < 0\}$  for all  $t > 0$ , which contradicts the fact that  $u(\cdot, t) : \mathbb{R}^n \rightarrow \mathbb{R}$  is smooth for  $t > 0$ .

So there exists  $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}^+$  with  $|\Delta u(x_0, t_0)|^2 \neq 0$ . Now  $u(cx, c^4t) = u(x, t)$  for all  $t > 0$ , for all  $x \in \mathbb{R}^n$  implies that (take the Laplacian w.r.t  $x$  of both sides)  $c^2(\Delta u)(cx, c^4t) = (\Delta u)(x, t)$  which implies  $|\Delta u|^2(cx, c^4t) = \frac{1}{c^4} |\Delta u|^2(x, t)$ . In particular, choosing  $t = t_0$ ,  $c^4 = (s/t_0)$  and  $x = x_0$  we find

$$|\Delta u|^2((s/t_0)^{1/4} x_0, s) = \frac{t_0}{s} |\Delta u|^2(x_0, t_0)$$

and hence  $|\Delta u|^2(x_s, s) = \frac{k_0}{s}$  where  $k_0 = t_0 |\Delta u(x_0, t_0)|^2 \neq 0$  and  $x_s = (s/t_0)^{1/4} x_0$ . Using an almost identical argument, we see that for all  $t > 0$ , there must be points  $y(t) \in \mathbb{R}^n$  such that  $(\Delta^2 u)(y(t), t) = \frac{k_1}{t}$  for some fixed  $k_1 \in \mathbb{R}$ ,  $k_1 \neq 0$ .

The fact that  $\frac{\partial}{\partial t} b = -\Delta^2 b$  can be seen as follows. Using Theorem 1.7 of Chapter I.1 in [SW], we have  $-|x|^4 e^{-|x|^4 t} = \frac{\partial}{\partial t} (e^{-|x|^4 t}) = (\frac{\partial}{\partial t} \widehat{b})(x, t) = \widehat{(\frac{\partial}{\partial t} b)}(x, t) = \widehat{(-\Delta^2 b)}(x, t)$ , and hence, taking the inverse of the Fourier transform, we get  $(\frac{\partial}{\partial t} b) = -\Delta^2 b$  (note that  $(\frac{\partial}{\partial t} \widehat{b})(x, t) = \widehat{(\frac{\partial}{\partial t} b)}(x, t)$  is true in view of the Lebesgue dominated convergence Theorem and the estimates (6.1) and the inverse of the Fourier transform exists in view of Corollary I.21 in I.1 of [SW] and the fact that  $b$  is in  $\mathcal{S}$ ). The fact that  $\int_{\mathbb{R}^n} b(z, 1) dz = 1$  may be seen by looking at how  $b$  was derived: Let  $u_0 : \mathbb{R}^n \rightarrow \mathbb{R}$  be a smooth function which is equal to 1 on  $B_1(0)$  and has compact support on  $B_2(0)$ . Hence  $u_0$  is in  $\mathcal{S}$ , and the Fourier transform  $\widehat{u_0}$  of  $u_0$  is also in  $\mathcal{S}$ . We only take the Fourier transform in the space direction in that which follows. Write  $u(x, t) = (b(\cdot, t) * u_0)(x)$  so  $\frac{\partial}{\partial t} u = -\Delta^2 u$  as explained above, and

$$(6.6) \quad \widehat{u}(x, t) = \widehat{b}(x, t) \widehat{u_0}(x) = e^{-t|x|^4} \widehat{u_0}(x)$$

(see Theorem 1.4 in I.1 of [SW]) and hence  $\widehat{u}(\cdot, t) \rightarrow \widehat{u_0}(\cdot)$  in the  $L^2$  sense as  $t \searrow 0$ . But this implies  $u(\cdot, t) \rightarrow u_0(\cdot)$  in the  $L^2$  sense as  $t \searrow 0$ , in view of the fact that the  $L^2$  norm is preserved for the Fourier transform (and the inverse Fourier transform) of a function in  $L^2 \cap L^1$  (see Theorem 2.1 and Theorem 2.4 in Chapter I.2 of [SW]).

In particular this shows  $\int_{\mathbb{R}^n} b = 1$ : If  $1 \neq c_0 := \int_{\mathbb{R}^n} b \neq 0$ , then for  $x \in B_{1/2}(0)$  we have

$$\begin{aligned}
|u(x, t) - 1/c_0| &= \left| \int_{\mathbb{R}^n} b(x - y, t)(u_0(y) - 1)dy \right| \\
&= \left| \int_{\mathbb{R}^n \setminus B_1(0)} b(x - y, t)(u_0(y) - 1)dy \right| \\
&= \left| \int_{\mathbb{R}^n \setminus B_1(0)} b(x - y, t)(u_0(y) - 1)dy \right| \\
&\leq C \int_{\mathbb{R}^n \setminus B_1(0)} |b(x - y, t)|dy \\
&= C \int_{\mathbb{R}^n \setminus B_1(x)} |b(z, t)|dz \\
&\leq C \int_{\mathbb{R}^n \setminus B_{1/2}(0)} |b(z, t)|dz \\
(6.7) \quad &\rightarrow 0
\end{aligned}$$

as  $t \searrow 0$  in view of (6.1), which shows  $u(\cdot, t)$  converges uniformly in the supremum norm to  $(1/c_0) \neq 1$  on  $B_{1/2}(0)$  as  $t \searrow 0$ , which contradicts the fact that  $u(\cdot, t)$  converges to  $u_0$  in the  $L^2$  norm as  $t \searrow 0$ . Similarly, if  $\int_{\mathbb{R}^n} b = 0$ , one shows  $u(x, t) \rightarrow 0$  uniformly in the supremum norm on  $B_{1/2}(0)$  as  $t \searrow 0$ , which contradicts the fact that  $u(\cdot, t)$  converges to  $u_0$  in the  $L^2$  norm as  $t \searrow 0$ .

## 7. APPENDIX

We require a rather specific form of the standard interpolation inequalities (see for example [GT, Theorems 7.25–7.28]). We have thus provided proofs and precise statements of that which we need here in the appendix.

**Lemma 7.1.** *For any smooth function  $\varphi : B_1(0) \rightarrow \mathbb{R}$  we have*

$$(7.1) \quad \int_{B_{1-s\sigma}} |\nabla^{2s}\varphi|^2 + |\nabla^{2s-1}\varphi|^2 \leq c(s, \sigma) \int_{B_1} |\Delta^s\varphi|^2 + |\Delta^{s-1}\varphi|^2 + \dots + |\varphi|^2$$

for any  $s \in \mathbb{N}$  and  $1 > \sigma > 0$ , as long as  $1 - s\sigma > 0$ .

*Proof.* We show the inequality (7.1) for arbitrary smooth  $\varphi : B_1(0) \rightarrow \mathbb{R}$  using induction.

Step 1:  $L^2$ -theory (see for example [E, Theorem 1, Section 6.3.1]) tells us that for an arbitrary smooth function  $\varphi : B_1 \rightarrow \mathbb{R}$

$$(7.2) \quad \int_{B_{1-\sigma}} |\nabla^2\varphi|^2 + |\nabla\varphi|^2 \leq c(\sigma) \int_{B_1} |\Delta\varphi|^2 + |\varphi|^2$$

as required.

Inductive Step: Let  $\alpha = (\alpha_1, \dots, \alpha_n)$  be a multi-index with  $0 \leq \alpha_i \leq 2s$  and  $\sum_{i=1}^n \alpha_i = 2s$ . We use the notation  $\nabla_\alpha \varphi := \nabla_{\alpha_1} \nabla_{\alpha_2} \dots \nabla_{\alpha_n} \varphi$ . Assume that statement (7.1) is true for some  $s \in \mathbb{N}$ . Again,  $L^2$  theory, see for example [E] Theorem



1, Section 6.3.1, tells us that

$$\begin{aligned}
& \int_{B_{1-(s+1)\sigma}} |\nabla^2 \nabla_\alpha \varphi|^2 + |\nabla \nabla_\alpha \varphi|^2 \\
& \leq a(s, \sigma) \int_{B_{1-s\sigma}} |\Delta(\nabla_\alpha \varphi)|^2 + |\nabla_\alpha \varphi|^2 \\
& = a(s, \sigma) \int_{B_{1-s\sigma}} |\nabla_\alpha(\Delta \varphi)|^2 + |\nabla_\alpha \varphi|^2 \\
& \leq a(s, \sigma) c(s, \sigma) \int_{B_1} |\Delta^s(\Delta \varphi)|^2 + |\Delta^{s-1}(\Delta \varphi)|^2 + \dots + |\Delta \varphi|^2 \\
& \quad + a(s, \sigma) c(s, \sigma) \int_{B_1} |\Delta^s \varphi|^2 + |\Delta^{s-1} \varphi|^2 + \dots + |\varphi|^2, \\
& = \tilde{a}(s, \sigma) \int_{B_1} |\Delta^{s+1} \varphi|^2 + |\Delta^s \varphi|^2 + \dots + |\varphi|^2
\end{aligned}$$

where in the second last line (the inequality) we have used the induction hypothesis applied to the functions  $\Delta \varphi$  and  $\varphi$ . By summing up over all possible  $\alpha$  (the number of possible  $\alpha$  is a constant depending on  $n$  and  $s$ ) we see that

$$\int_{B_{1-(s+1)\sigma}} |\nabla^{2s+2} \varphi|^2 + |\nabla^{2s+1} \varphi|^2 \leq c(s+1, \sigma) \int_{B_1} |\Delta^{s+1} \varphi|^2 + |\Delta^s \varphi|^2 + \dots + |\varphi|^2$$

as required.

This completes the proof by induction.  $\square$

**Lemma 7.2.** *Suppose  $u \in C_{loc}^\infty(\mathbb{R}^n)$ ,  $k \in \mathbb{N}$ ,  $s > 8$ , and  $\gamma, c_\gamma$  are as in  $(\gamma)$ . For any  $\delta_0 > 0$  we have*

$$\int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} \leq \delta_0 \rho^4 \int_{\mathbb{R}^n} |\Delta^{k+1} u|^2 \gamma^s + \frac{c_{\delta_0}}{\rho^4} \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8}.$$

where  $c_{\delta_0}$  is an absolute constant given by

$$c_{\delta_0} = c_{\delta_0}(\delta_0, s, n) = 2^4(\delta_0^{-1} + 2^9 s^4 c_\gamma^4).$$

*Proof.* Integrating by parts,

$$\begin{aligned}
\int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} &= - \int_{\mathbb{R}^n} (\nabla_i \Delta^k u) (\nabla_i \Delta^{k-1} u) \gamma^{s-4} \\
&\quad - (s-4) \int_{\mathbb{R}^n} (\Delta^k u) (\nabla_i \Delta^{k-1} u) (\nabla_i \gamma) \gamma^{s-5} \\
&= \int_{\mathbb{R}^n} (\Delta^{k+1} u) (\Delta^{k-1} u) \gamma^{s-4} \\
&\quad + (s-4) \int_{\mathbb{R}^n} (\nabla_i \Delta^k u) (\Delta^{k-1} u) (\nabla_i \gamma) \gamma^{s-5} \\
&\quad - (s-4) \int_{\mathbb{R}^n} (\Delta^k u) (\nabla_i \Delta^{k-1} u) (\nabla_i \gamma) \gamma^{s-5} \\
&= \int_{\mathbb{R}^n} (\Delta^{k+1} u) (\Delta^{k-1} u) \gamma^{s-4} \\
&\quad - 2(s-4) \int_{\mathbb{R}^n} (\Delta^k u) (\nabla_i \Delta^{k-1} u) (\nabla_i \gamma) \gamma^{s-5} \\
&\quad - (s-4) \int_{\mathbb{R}^n} (\Delta^k u) (\Delta^{k-1} u) (\Delta \gamma) \gamma^{s-5} \\
&\quad - (s-4)(s-5) \int_{\mathbb{R}^n} (\Delta^k u) (\Delta^{k-1} u) |\nabla \gamma|^2 \gamma^{s-6}.
\end{aligned}$$

Estimating the right hand side with Young's inequality (estimate (3.3)), we have for any  $\delta_i > 0$

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} \\
&\leq \delta_1 \rho^4 \int_{\mathbb{R}^n} |\Delta^{k+1} u|^2 \gamma^s + \frac{1}{4\delta_1 \rho^4} \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8} \\
&\quad + \delta_2 \int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} + \frac{c_\gamma^2 (s-4)^2}{\delta_2 \rho^2} \int_{\mathbb{R}^n} |\nabla \Delta^{k-1} u|^2 \gamma^{s-6} \\
&\quad + \delta_3 \int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} + \frac{c_\gamma^2 (s-4)^2}{4\delta_3 \rho^4} \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-6} \\
&\quad + \delta_4 \int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} + \frac{c_\gamma^4 (s-4)^2 (s-5)^2}{4\delta_4 \rho^4} \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8},
\end{aligned}$$

which upon absorption gives

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} \\
&\leq 4\delta_1 \rho^4 \int_{\mathbb{R}^n} |\Delta^{k+1} u|^2 \gamma^s + \frac{16c_\gamma^2 (s-4)^2}{\rho^2} \int_{\mathbb{R}^n} |\nabla \Delta^{k-1} u|^2 \gamma^{s-6} \\
(7.3) \quad &\quad + \frac{1}{\rho^4} \left( \frac{1}{\delta_1} + 4c_\gamma^2 (s-4)^2 + 4c_\gamma^4 (s-4)^2 (s-5)^2 \right) \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8},
\end{aligned}$$

where we have chosen  $\delta_2 = \delta_3 = \delta_4 = \frac{1}{4}$ , and we used  $\gamma^{s-6}(\cdot) \leq \gamma^{s-8}(\cdot)$ , which follows in view of  $0 \leq \gamma \leq 1$  and  $s-6 > s-8 \geq 1$ . For the second term we integrate

by parts and estimate using Young's inequality to obtain

$$\begin{aligned}
& \int_{\mathbb{R}^n} |\nabla \Delta^{k-1} u|^2 \gamma^{s-6} \\
&= - \int_{\mathbb{R}^n} (\Delta^k u) (\Delta^{k-1} u) \gamma^{s-6} - (s-6) \int_{\mathbb{R}^n} (\Delta^{k-1} u) (\nabla_i \Delta^{k-1} u) (\nabla_i \gamma) \gamma^{s-7} \\
&\leq \frac{\rho^2}{64c_\gamma^2(s-4)^2} \int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} + \frac{16c_\gamma^2(s-4)^2}{\rho^2} \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8} \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla \Delta^{k-1} u|^2 \gamma^{s-6} + \frac{c_\gamma^2(s-6)^2}{2\rho^2} \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8}.
\end{aligned}$$

Absorbing the third term on the right into the left yields

$$\begin{aligned}
& \int_{\mathbb{R}^n} |\nabla \Delta^{k-1} u|^2 \gamma^{s-6} \leq \frac{\rho^2}{32c_\gamma^2(s-4)^2} \int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} \\
&\quad + \frac{c_\gamma^2}{\rho^2} \left( 32(s-4)^2 + (s-6)^2 \right) \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8},
\end{aligned}$$

and so

$$\begin{aligned}
& \frac{16c_\gamma^2(s-4)^2}{\rho^2} \int_{\mathbb{R}^n} |\nabla \Delta^{k-1} u|^2 \gamma^{s-6} \leq \frac{1}{2} \int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} \\
&\quad + \frac{16c_\gamma^4(s-4)^2}{\rho^4} \left( 32(s-4)^2 + (s-6)^2 \right) \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8}.
\end{aligned}$$

Combining the above with (7.3) we have

$$\begin{aligned}
& \int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} \leq 4\delta_1 \rho^4 \int_{\mathbb{R}^n} |\Delta^{k+1} u|^2 \gamma^s + \frac{1}{2} \int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} \\
&\quad + \frac{1}{\rho^4} \left( 16c_\gamma^4(s-4)^2 \left( 32(s-4)^2 + (s-6)^2 \right) \right. \\
&\quad \left. + \frac{1}{\delta_1} + 4c_\gamma^2(s-4)^2 + 4c_\gamma^4(s-4)^2(s-5)^2 \right) \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8},
\end{aligned}$$

Absorbing the second term on the right into the left, multiplying through by 2 and choosing  $\delta_1 = \frac{\delta_0}{8}$  yields the estimate

$$\int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} \leq \delta_0 \rho^4 \int_{\mathbb{R}^n} |\Delta^{k+1} u|^2 \gamma^s + \frac{\tilde{c}_{\delta_0}}{\rho^4} \int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8}$$

where

$$\begin{aligned}
\tilde{c}_{\delta_0} &= 32c_\gamma^4(s-4)^2 \left( 32(s-4)^2 + (s-6)^2 \right) + \frac{16}{\delta_0} + 8(s-4)^2 (c_\gamma^2 + c_\gamma^4(s-5)^2) \\
&= \frac{16}{\delta_0} + 8(s-4)^2 c_\gamma^2 \left( 1 + c_\gamma^2(s-5)^2 + 4c_\gamma^2 \left( 32(s-4)^2 + (s-6)^2 \right) \right).
\end{aligned}$$

Since  $s > 8$  and  $c_\gamma \geq 1$  we have

$$\begin{aligned}\tilde{c}_{\delta_0} &\leq \frac{16}{\delta_0} + 8s^2 c_\gamma^2 (1 + c_\gamma^2 s^2 + 4c_\gamma^2 (32s^2 + s^2)) \\ &\leq \frac{2^4}{\delta_0} + 2^3 s^4 c_\gamma^2 (1 + c_\gamma^2 + 2^2 c_\gamma^2 (2^5 + 1)) \\ &\leq 2^4 (\delta_0^{-1} + 2^9 s^4 c_\gamma^4) \\ &:= c_{\delta_0},\end{aligned}$$

yielding the constant claimed.  $\square$

**Corollary 7.3.** *Suppose  $u \in C_{loc}^\infty(\mathbb{R}^n)$ ,  $k \in \mathbb{N}$ , and  $\gamma$ ,  $c_\gamma$  are as in (7). For all  $k \in \mathbb{N}$ ,  $s > 4k$ ,*

$$\int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} \leq \delta \rho^4 \int_{\mathbb{R}^n} |\Delta^{k+1} u|^2 \gamma^s + \frac{c}{\rho^{4k-4}} \int_{\mathbb{R}^n} |\Delta u|^2 \gamma^{s-4k},$$

where  $c(\delta, s, n) < \infty$  is a constant depending on  $\delta, s, n$ .

*Proof.* We shall proceed by induction in  $k \in \mathbb{N}$ . We wish to show that

$$(7.4) \quad E_{\gamma^{s-4}}^k(u) \leq \hat{\delta} \rho^4 E_{\gamma^s}^{k+1}(u) + \frac{\hat{c}_{\delta,k,s}}{\rho^{4k-4}} E_{\gamma^{s-4k}}^1(u).$$

is true for all  $s > 4k$  for all  $\hat{\delta} > 0$ , where  $\hat{c}_{\delta,k,s} = \hat{c}(\hat{\delta}, s, k, n)$ . Let  $k = 1$ . Then (7.4) is true for all  $s > 4k$  for all  $\delta > 0$  trivially with the choice of  $\hat{c}_{\delta,k,s} = 1$ .

Assume (7.4) for some fixed  $k \in \mathbb{N}$ , and let  $s > 4(k+1)$ ,  $\delta > 0$  be arbitrary. Then  $s > 8$  and Lemma 7.2 gives the estimate

$$(7.5) \quad E_{\gamma^{s-4}}^l(u) \leq \delta \rho^4 E_{\gamma^s}^{l+1}(u) + \frac{c_{\delta,l,s}}{\rho^4} E_{\gamma^{s-8}}^{l-1}(u)$$

for any  $l \in \mathbb{N}$ , where  $c_{\delta,l,s} = c(\delta, l, s, n)$ .

Using this inequality with  $l = k+1 \in \mathbb{N}$  and then (7.4) we have

$$\begin{aligned}E_{\gamma^{s-4}}^{k+1}(u) &\leq \delta \rho^4 E_{\gamma^s}^{k+2}(u) + \frac{c_{\delta,k+1,s}}{\rho^4} E_{\gamma^{s-8}}^k(u) \\ &\leq \delta \rho^4 E_{\gamma^s}^{k+2}(u) + \hat{\delta} c_{\delta,k+1,s} E_{\gamma^{s-4}}^{k+1}(u) + \frac{\hat{c}_{\delta,k,s-4} c_{\delta,k+1,s}}{\rho^{4k}} E_{\gamma^{s-4-4k}}^1(u).\end{aligned}$$

where here we used the fact that  $s > 4(k+1) = 4k+4$  implies that  $\tilde{s} = s-4 > 4k$  and so (7.4) is valid with  $s$  replaced by  $\tilde{s}$ . Choosing  $\hat{\delta} c_{\delta,k+1,s} = \frac{1}{2}$  and absorbing we obtain

$$E_{\gamma^{s-4}}^{k+1}(u) \leq 2\delta \rho^4 E_{\gamma^s}^{k+2}(u) + 2 \frac{\hat{c}_{\tilde{s},k,s-4} c_{\delta,k+1,s}}{\rho^{4k}} E_{\gamma^{s-4-4k}}^1(u).$$

which gives us the result, as  $\delta > 0$  was arbitrary. Note that we can ensure the constant only depends on  $n, s$  and not  $k$  by taking the supremum of the constants we obtained in this argument, where this supremum is taken over all  $4k < s$ ,  $k \in \mathbb{N}$ , for a fixed  $s \in \mathbb{N}$ .  $\square$

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